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General Jacobi Identity Revisited Again

Gröbner Bases in Differential Geometry

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Abstract Synthetic differential geometry occupies a unique position in topos-theoretic physics. Nevertheless it has appeared somewhat too conceptual to physicists in general, partly because it has appeared to lack computational aspects. Its computational facets are really concerned with computation of the quasi-colimit of a finite diagram of infinitesimal spaces, or equivalently, with computation of the limit of a finite diagram of Weil algebras. Indeed we have been forced to do a highly involved computation of the above kind by hand in our previous papers (Nishimura, H. in Int. J. Theor. Phys. 36:1099–1131, 1997 and Nishimura, H. in Int. J. Theor. Phys. 38:2163–2174, 1999). The principal objective in this paper is to show that Gröbner bases techniques provide us with means that relegate such computations to computers.

Keywords Weil algebra · Limit · Equalizer · Gröbner basis · Elimination theory · Synthetic differential geometry · Zero-dimensional ideal · Topos-theoretic physics

1 Introduction

Synthetic differential geometry, in which nilpotent infinitesimals are visible geometrically, is expected to play a predominant role in topos-theoretic physics (cf. [4]). Nevertheless it has appeared somewhat too conceptual to physicists in general. Physicists prefer computations, while synthetic differential geometers have so far made little account of computational aspects. In using orthodox differential geometry, physicists enjoy computation in local charts, while the main objects of study in synthetic differential geometry are microlinear spaces, in which coordinates are generally out of touch. The lacking of coordinates make physicists feel somewhat alienated, but we should stress that computation of the quasi-colimit of infinitesimal spaces, or equivalently, computation of the limit of Weil algebras play the same

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role in synthetic differential geometry as computation in local charts does in orthodox differential geometry. Even such computational aspects in synthetic differential geometry have appeared still too conceptual to many poor physicists. The principal objective in this paper is to show that they can be relegated to computers.

We denote by $\mathbb{R}$ the set of real numbers. The category of $\mathbb{R}$-algebras and their homomorphisms is denoted by $\mathbb{R}-\mathbf{Alg}$, while its full subcategory of finitely presentable $\mathbb{R}$-algebras is denoted by $\mathbb{R}-\mathbf{Alg}_{\mathrm{fp}}$. Since we deal exclusively with *commutative* algebras in this paper, commutativity is not explicitly stated. It is well known (cf. [5, Appendix A]) that

Theorem 1.1 *Let* $\mathbf{E}$ *be a category with finite inverse limits and an* $\mathbb{R}$*-algebra object* $\mathbf{R}$. *Then there exists, up to isomorphisms, a unique functor* $\mathrm{Spec}_{\mathbf{R}}:(\mathbb{R}-\mathbf{Alg}_{\mathrm{fp}})^{\mathrm{op}}\to\mathbf{E}$ *preserving finite inverse limits and taking* $\mathbb{R}[x]$ *to* $\mathbf{R}$.

A *Weil algebra* (*over* $\mathbb{R}$) is a finite-dimensional $\mathbb{R}$-algebra $\mathfrak{W}$ subject to the following conditions:

(1.1) $\mathfrak{W}$ is a local ring in the sense that for all $a,b\in\mathfrak{W}$, if $a+b=1$, then either a or b is invertible.

(1.2) $\mathfrak{W}$ can be written as $\mathfrak{W}=\mathbb{R}\oplus\mathfrak{m}$, where the first component is the $\mathbb{R}$-algebra structure, and the second is the maximal ideal of $\mathfrak{W}$.

The reader should note that $\mathfrak{m}$ is nilpotent by Nakayama's lemma, for which he or she is referred to Corollary 3.16 of [9, Chapter I]. For some characterizations of Weil algebras, the reader is referred to Theorem 3.17 of [9, Chapter I]. In particular, a Weil algebra $\mathfrak{W}$ is representable as an affine $\mathbb{R}$-algebra $\mathbb{R}[x_1,\dots,x_n]/I$, in which there exists a natural number k with $x_i^k\in I$ for any indeterminate x_i.

A *homomorphism of Weil algebras* from a Weil algebra $\mathfrak{W}_1$ to another (possibly the same) Weil algebra $\mathfrak{W}_2$ is a homomorphism of $\mathbb{R}$-algebras from $\mathfrak{W}_1$ to $\mathfrak{W}_2$ mapping the maximal ideal of $\mathfrak{W}_1$ into the maximal ideal of $\mathfrak{W}_2$. We denote by **Wl** the category of Weil algebras and homomorphisms of Weil algebras, which is a subcategory of the category $\mathbb{R}-\mathbf{Alg}$. It is well known that the category $\mathbb{R}-\mathbf{Alg}$ is finitely complete. In Sect. 2 we will show that the category **Wl** is also finitely complete. We will give a necessary and sufficient condition for the coincidence of the limit of a given diagram in **Wl** with that in $\mathbb{R}-\mathbf{Alg}$. The limit of a diagram in **Wl** is called *good* if it happens to be the limit of the same diagram in $\mathbb{R}-\mathbf{Alg}$. The notion of a good limit is important, because it is well known (cf. [6, Proposition 1.2]) that

Proposition 1.2 *Let* $\mathbf{E}$ *be a Cartesian closed category with finite inverse limits and an* $\mathbb{R}-$*algebra object* $\mathbf{R}$. *Let us assume that* $\mathbf{E}$ *and* $\mathbf{R}$ *satisfy the following Kock–Lawvere axiom*:

(1.3) *For any Weil algebra* $\mathfrak{W}$, *the natural* $\mathbf{R}$*-algebra homomorphism*

$$\mathbf{R}\otimes\mathfrak{W}\to\mathbf{R}^{\mathrm{Spec}_{\mathbf{R}}(\mathfrak{W})}$$

is an isomorphism.

Then the functor $\mathbf{R}^{\mathrm{Spec}_{\mathbf{R}}(\cdot)}:\mathbf{Wl}\to\mathbf{E}$ *takes good finite limit diagrams to limit diagrams. Conversely, if a finite diagram in* **Wl** *is taken to a limit diagram in* $\mathbf{E}$ *by the functor* $\mathbf{R}^{\mathrm{Spec}_{\mathbf{R}}(\cdot)}$, *then the diagram is a good limit diagram, provided that* $\Gamma(\mathbf{R})=\mathbb{R}$, *where* Γ *is the global sections functor.*

In synthetic differential geometry, the notion of a smooth manifold in orthodox differential geometry should be replaced by that of a microlinear space. Just as local charts enable orthodox differential geometers to transfer from $\mathbb{R}$ to smooth manifolds, good finite limit diagrams in **Wl** enable synthetic differential geometers to transfer from **R** to microlinear spaces. This makes the study of Weil algebras downright central in synthetic differential geometry. For textbooks on synthetic differential geometry, the reader is referred to [5, 7, 9].

The principal objective in Sect. 3 is to give an efficient algorithm for calculating finite limits in **Wl**. The calculation of limits of finite diagrams in any finitely complete category is reducible to the calculation of products and equalizers, because the limit of a finite diagram is representable in terms of products and equalizers, for which the reader is referred to [8, Chap. V, §2]. Here the calculation of the limit of a finite diagram in **Wl** means exactly the presentation of the limit of the diagram as an affine $\mathbb{R}$-algebra, provided that each object in the diagram is presented as an affine $\mathbb{R}$-algebra. The calculation of the product of two Weil algebras in the category **Wl** is simple enough. If the two Weil algebras are presented by $\mathbb{R}[x_1, \ldots, x_m]/I$ and $\mathbb{R}[y_1, \ldots, y_n]/J$ as affine $\mathbb{R}$-algebras, then their product is represented by $\mathbb{R}[x_1, \ldots, x_m, y_1, \ldots, y_n]/K$, where K is the ideal generated by I, J and $x_i y_j' s$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$. The calculation of equalizers in **Wl** is not so simple. We must solve linear equations and evoke the elimination theory, in which familiar Gröbner bases techniques should be invoked. Our standard reference on the theory of Gröbner bases is [1].

Fortunately or unfortunately, synthetic differential geometry has dealt with so simple diagrams that synthetic differential geometers have not felt the need of such algorithms keenly. This is why they have been content with their naively combinatorial arguments. The remarkable exception is [10–12] general Jacobi identity, for which a moderately involved diagram should be considered. The limit diagram for the general Jacobi identity lies on the verge of feasibility of naively combinatorial arguments, which is why the general Jacobi identity, though being fundamental in synthetic differential geometry and even in mathematics, had remained to be discovered for so long. We hope that our new algorithm will transmogrify the landscape drastically in manipulation of diagrams in **Wl**, just as the theory of linear equations has made the calculation of the value of two unknown quantities from their unit total and the total of one of their attributes everyone's job. It is generally believed that zero-dimensional ideals render an attractive forum to the theory of Gröbner bases, and we hold that the theory of Weil algebras and the study of zero-dimensional ideals can and should weave together.

Section 2 is devoted to a short course on the category of. In Sects. 4–6 we deal with three examples. The examples of Sects. 4 and 5 are so simple that the reader might feel that we are pedantic and frothy enough to brandish and trot out our new algorithm in such simple cases. We contend that the best way to understand a new algorithm is to apply the algorithm to some well understood examples. The last example dealt with in Sect. 6 is moderately involved. To deal with such an example without our algorithm is to calculate the respective numbers of cranes and tortoises from the totals of their heads and legs without knowing the theory of linear equations at all. It is feasible as we did in [10, 11], but the spirit of algebra and algorithm traced back to medieval Arabians has encouraged us to find out a more systematic way. For the geometrical background of Sects. 4 and 5, the reader is referred to [3, 6]. For the geometric background of Sect. 6, the reader is referred to [10–12].

2 The Category of Weil Algebras

The category $\mathbb{R}-\mathbf{Alg}$ is well known to be finitely complete. We would like to show that the category $\mathbf{Wl}$ is also finitely complete. It is easy to see that

Proposition 2.1 *$\mathbb{R}$ is a terminal object in the category* $\mathbf{Wl}$.

Given two Weil algebras $\mathfrak{W}_1=\mathbb{R}\oplus\mathfrak{m}_1$ and $\mathfrak{W}_2=\mathbb{R}\oplus\mathfrak{m}_2$, their product is defined as $\mathbb{R}\oplus\mathfrak{m}_1\oplus\mathfrak{m}_2$, in which the multiplication in $\mathfrak{W}_1=\mathbb{R}\oplus\mathfrak{m}_1$ and $\mathfrak{W}_2=\mathbb{R}\oplus\mathfrak{m}_2$ persists, while it is stipulated that the multiplication of any element of $\mathfrak{m}_1$ and any element of $\mathfrak{m}_2$ should vanish. The product is denoted by $\mathfrak{W}_1\underline{\oplus}\mathfrak{W}_2$. The mapping $(a,m_1,m_2)\in\mathbb{R}\oplus\mathfrak{m}_1\oplus\mathfrak{m}_2\mapsto(a,m_1)\in\mathfrak{W}_1$ is denoted by π_1, while the mapping $(a,m_1,m_2)\in\mathbb{R}\oplus\mathfrak{m}_1\oplus\mathfrak{m}_2\mapsto(a,m_2)\in\mathfrak{W}_2$ is denoted by π_2. Then it is easy to see that

Proposition 2.2 *For any Weil algebras $\mathfrak{W}_1$ and $\mathfrak{W}_2$, the diagram*

$$\mathfrak{W}_1\xleftarrow{\pi_1}\mathfrak{W}_1\underline{\oplus}\mathfrak{W}_2\xrightarrow{\pi_2}\mathfrak{W}_2$$

is a product of $\mathfrak{W}_1$ and $\mathfrak{W}_2$ in the category $\mathbf{Wl}$.

Note that, given two arrows $\varphi:\mathfrak{W}_1\to\mathfrak{W}_1'$ and $\psi:\mathfrak{W}_2\to\mathfrak{W}_2'$ in $\mathbf{Wl}$, there exists a unique arrow $\varphi\underline{\oplus}\psi:\mathfrak{W}_1\underline{\oplus}\mathfrak{W}_2\to\mathfrak{W}_1'\underline{\oplus}\mathfrak{W}_2'$ in $\mathbf{Wl}$ such that $\pi_1\circ(\varphi\underline{\oplus}\psi)=\varphi\circ\pi_1$ and $\pi_2\circ(\varphi\underline{\oplus}\psi)=\psi\circ\pi_2$.

It is also easy to see that

Proposition 2.3 *For any parallel arrows $\mathfrak{W}_1\rightrightarrows\mathfrak{W}_2$ in the category* $\mathbf{Wl}$, *its equalizer $\mathfrak{W}_\infty\to\mathfrak{W}_1\rightrightarrows\mathfrak{W}_2$ in the category $\mathbb{R}-\mathbf{Alg}$ belongs in its subcategory* $\mathbf{Wl}$.

Theorem 2.4 *The category* $\mathbf{Wl}$ *is finitely complete.*

Proof This follows simply from Propositions 2.1–2.3 by dint of Theorem 1 of [8, Chap. V, §2]. □

Let $U_1:\mathbf{Wl}\to\mathbb{R}-\mathbf{Alg}$and $U_2:\mathbb{R}-\mathbf{Alg}\to\mathbb{R}-\mathbf{Mod}$ be forgetful functors, where $\mathbb{R}-\mathbf{Mod}$ is the category of linear spaces over $\mathbb{R}$ and their linear maps. Let $M:\mathbf{Wl}\to\mathbb{R}-\mathbf{Mod}$ be a functor assigning, to each Weil algebra, its maximal ideal and, to each homomorphism of Weil algebras, its restriction to maximal ideals. Every category is uniquely decomposed into connected categories, each of which is called a *connected component*. For the definition of the connectedness of a category, the reader is referred to [2, 2.6.7.e], or [13, Definition 9.1.1].

Theorem 2.5 *Let $F:\mathbf{J}\to\mathbf{Wl}$ be a finite diagram of Weil algebras. Then $U_1(\underleftarrow{\mathrm{Lim}}\,F)$ and $\underleftarrow{\mathrm{Lim}}\,U_1\circ F$ are naturally isomorphic iff the category $\mathbf{J}$ is connected.*

Proof It suffices to show that $U_2\circ U_1(\underleftarrow{\mathrm{Lim}}\,F)$ and $U_2(\underleftarrow{\mathrm{Lim}}\,U_1\circ F)$ are naturally isomorphic. It is easy to see that

$$U_2\circ U_1(\underleftarrow{\mathrm{Lim}}\,F)=\mathbb{R}+\underleftarrow{\mathrm{Lim}}\,M\circ F \tag{2.4}$$

while

$$U_2(\underset{\longleftarrow}{\mathrm{Lim}}\, U_1 \circ F) = \mathbb{R}^m + \underset{\longleftarrow}{\mathrm{Lim}}\, M \circ F, \tag{2.5}$$

where $m$is the number of connected components of **J**. Therefore the desired statement follows. □

3 Algorithms

In order to calculate finite limits in **Wl**, it suffices to calculate finite products and equalizers in **Wl**. The calculation of finite products in **Wl** is straightforward. Given two Weil algebras $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as affine $\mathbb{R}$-algebras $\mathbb{R}[x_1,\ldots,x_n]/I$ and $\mathbb{R}[y_1,\ldots,y_m]/J$ respectively, $\mathfrak{W}_1 \underline{\oplus} \mathfrak{W}_2$ is represented as an affine algebra $\mathbb{R}[x_1,\ldots,x_n,y_1,\ldots,y_m]/K$, where K is the ideal generated by I, J and $x_i y_j$ for all natural numbers i, j with $1 \le i \le n$ and $1 \le j \le m$.

We will present an algorithm for calculating equalizers. Given parallel arrows $\mathfrak{W}_1 \overset{\varphi}{\underset{\psi}{\rightrightarrows}} \mathfrak{W}_2$ in **Wl** with $\mathfrak{W}_1 = \mathbb{R} \oplus \mathfrak{m}_1$ and $\mathfrak{W}_2 = \mathbb{R} \oplus \mathfrak{m}_2$, let $\mathsf{e}_1,\ldots,\mathsf{e}_p$ and $\mathsf{f}_1,\ldots,\mathsf{f}_q$ be linear bases of $\mathfrak{m}_1$ and $\mathfrak{m}_2$ respectively. Since $\varphi(\mathfrak{m}_1) \subset \mathfrak{m}_2$ and $\psi(\mathfrak{m}_1) \subset \mathfrak{m}_2$, there exist $a_i^j, b_i^j \in \mathbb{R}$ ($1 \le i \le p$ and $1 \le j \le q$) such that

$$\varphi(\mathsf{e}_i) = \sum_{j=1}^{q} a_i^j \mathsf{f}_j, \tag{3.1}$$

$$\psi(\mathsf{e}_i) = \sum_{j=1}^{q} b_i^j \mathsf{f}_j. \tag{3.2}$$

By solving the linear equation

$$\begin{bmatrix} a_1^1 & \cdots & a_p^1 \\ \vdots & & \vdots \\ a_1^q & \cdots & a_p^q \end{bmatrix} \begin{bmatrix} x^1 \\ \vdots \\ x^p \end{bmatrix} = \begin{bmatrix} b_1^1 & \cdots & b_p^1 \\ \vdots & & \vdots \\ b_1^q & \cdots & b_p^q \end{bmatrix} \begin{bmatrix} x^1 \\ \vdots \\ x^p \end{bmatrix} \tag{3.3}$$

we get $(c_1^1,\ldots c_1^p),\ldots,(c_r^1,\ldots c_r^p)$ as a system of fundamental solutions of (3.3). We let g_k be

$$\mathsf{g}_k = \sum_{i=1}^{p} c_k^i \mathsf{e}_i \quad (1 \le k \le r). \tag{3.4}$$

Therefore, given $\mathsf{x} \in \mathfrak{W}_1$, we have $\varphi(\mathsf{x}) = \psi(\mathsf{x})$ iff it is a linear combination of $1,\mathsf{g}_1,\ldots,\mathsf{g}_r$. Thus we get an equalizer $\mathfrak{W}_\infty = \mathbb{R} \oplus \mathfrak{m}_\infty$, where $\mathfrak{m}_\infty$ is generated linearly by $\mathsf{g}_1,\ldots,\mathsf{g}_r$.

Our story of equalizers in **Wl** is not over. Although $\mathsf{g}_1,\ldots,\mathsf{g}_r$ are linearly independent by definition so that there is no linear redundance among $\mathsf{g}_1,\ldots,\mathsf{g}_r$, it may be the case, by way of example, that g_1 is representable as a polynomial of $\mathsf{g}_2,\ldots,\mathsf{g}_r$ so that g_1 is $\mathbb{R}$-algebraically redundant. Therefore we must find out a subset $\mathsf{g}_{k_1},\ldots,\mathsf{g}_{k_s}$ of $\mathsf{g}_1,\ldots,\mathsf{g}_r$, for which there is no such $\mathbb{R}$-algebraic redundance. Once we find such a subset $\mathsf{g}_{k_1},\ldots,\mathsf{g}_{k_s}$ of $\mathsf{g}_1,\ldots,\mathsf{g}_r$, we have to find out the ideal of relations among $\mathsf{g}_{k_1},\ldots,\mathsf{g}_{k_s}$.

We assume throughout the rest of this section that $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are finitely presented as affine $\mathbb{R}$-algebras $\mathbb{R}[x_1,\ldots,x_n]/I$ and $\mathbb{R}[y_1,\ldots,y_m]/J$ respectively. Let us start on the first task.

Theorem 3.1 *Let $\mathfrak{g}_1,\ldots,\mathfrak{g}_r$ be polynomials in $\mathbb{R}[x_1,\ldots,x_n]$. Let $l\le r$. Let I be an ideal in the $\mathbb{R}$-algebra $\mathbb{R}[x_1,\ldots,x_n]$. Let $\mathcal{G}$ be the reduced Gröbner basis for the ideal $\langle I, w_1-\mathfrak{g}_1,\ldots,w_r-\mathfrak{g}_r\rangle$ in the $\mathbb{R}$-algebra $\mathbb{R}[w_1,\ldots,w_r,x_1,\ldots,x_n]$ with respect to an elimination order with the x variables larger than the w variables and, at the same time, the variables $w_{l+1},\ldots,w_r$ larger than the variables $w_1,\ldots,w_l$ (e.g., the lexicographic order $w_1<\cdots<w_l<w_{l+1}<\cdots<w_r<x_1<\cdots<x_n$). Exactly speaking about the elimination order, we have, for any power products X, X' in variables $x_1,\ldots,x_n$, any power products W_1, W_1' in variables $w_{l+1},\ldots,w_r$ and any power products W_2, W_2' in variables $w_1,\ldots,w_l$,*

$$XW_1W_2 < X'W_1'W_2' \quad \textit{iff} \quad \left\{\begin{array}{c} X<X' \\ X=X' \textit{ and } W_1<W_1' \\ \textit{or} \\ X=X',\, W_1=W_1' \textit{ and } W_2<W_2'. \end{array}\right\}. \tag{3.5}$$

Then the $\mathbb{R}$-subalgebra $\mathfrak{V}$ of $\mathbb{R}[x_1,\ldots,x_n]/I$ generated by $\mathfrak{g}_1+I,\ldots,\mathfrak{g}_r+I$ is already generated by $\mathfrak{g}_1+I,\ldots,\mathfrak{g}_l+I$ iff $\mathcal{G}\cap\mathbb{R}[w_1,\ldots,w_r]$ contains a polynomial $w_k-\mathfrak{h}_k$ with $\mathfrak{h}_k\in\mathbb{R}[w_1,\ldots,w_l]$ $(l+1\le k\le r)$. In this case, $\mathcal{G}\cap\mathbb{R}[w_1,\ldots,w_l]$ is a Gröbner basis for the ideal of relations among $w_1,\ldots,w_l$ with w_k representing $\mathfrak{g}_k+I$ $(1\le k\le l)$, and we have $\mathfrak{g}_k+I=\mathfrak{h}_k(\mathfrak{g}_1+I,\ldots,\mathfrak{g}_l+I)$ $(l+1\le k\le r)$.

Proof In order to get the first conclusion in the theorem, it suffices to apply the discussion in the proofs of Theorems 2.4.7 and 2.4.13 of [1] to the $\mathbb{R}$-algebra homomorphism from $\mathbb{R}[w_1,\ldots,w_l]$ to $\mathbb{R}[w_{l+1},\ldots,w_k,x_1,\ldots,x_n]/K$ assigning $\mathfrak{g}_k+K$ to w_k $(1\le k\le l)$, where K is the ideal $\langle I, w_{l+1}-\mathfrak{g}_{l+1},\ldots,w_r-\mathfrak{g}_r\rangle$ in the $\mathbb{R}$-algebra $\mathbb{R}[w_{l+1},\ldots,w_k,x_1,\ldots,x_n]$, where it is established in the course that we have $\mathfrak{g}_k+I=\mathfrak{h}_k(\mathfrak{g}_1+I,\ldots,\mathfrak{g}_l+I)$ $(l+1\le k\le r)$ in this case. That if one of the equivalent conditions in the first conclusion holds, then $\mathcal{G}\cap\mathbb{R}[w_1,\ldots,w_l]$ is a Gröbner basis for the ideal of relations among $w_1,\ldots,w_l$ with w_k representing $\mathfrak{g}_k+I$ $(1\le k\le l)$ follows by Theorems 2.3.4 and 2.4.10 of [1]. □

This theorem gives the following algorithm for our first task.

Algorithms 3.2 (Weeding out $\mathbb{R}$-algebraic redundance)

Input: arbitrary generators $\mathsf{g}_1,\ldots,\mathsf{g}_r$ of the $\mathbb{R}$-algebra $\mathfrak{W}_\infty$.

Output: generators $\mathsf{g}_{k_1},\ldots,\mathsf{g}_{k_s}$ of $\mathfrak{W}_\infty$ without $\mathbb{R}$-algebraic redundance.

1. Let $X=\{\mathsf{g}_1,\ldots,\mathsf{g}_r\}$ and $Y=\phi$ (the empty set).

2. Choose g_k from X and ask whether g_k is in the $\mathbb{R}$-subalgebra generated by $(X-\{\mathsf{g}_k\})\cup Y$, which is decidable by Theorem 3.1. If the answer is yes, then let $X:=X-\{\mathsf{g}_k\}$. If the answer is no, then let $X:=X-\{\mathsf{g}_k\}$ and $Y:=Y\cup\{\mathsf{g}_k\}$. Repeat this process until X becomes empty.

3. Output Y.

Now we turn to our second task. This can be carried out by the following familiar algorithm, for which the reader is referred to Theorem 2.3.4 and Theorem 2.4.10 of [1].

Algorithm 3.3 (Computing a Gröbner basis for the ideal of relations)

Input: generators $\mathsf{g}_{k_1}, \ldots, \mathsf{g}_{k_s}$ of the $\mathbb{R}$-algebra $\mathfrak{W}_\infty$.

Output: a Gröbner basis for the ideal of relations among $\mathsf{g}_{k_1}, \ldots, \mathsf{g}_{k_s}$.

1. Compute a Gröbner basis $\mathcal{G}$ for the ideal $\langle I, w_1 - \mathfrak{g}_{k_1}, \ldots, w_s - \mathfrak{g}_{k_s} \rangle$ in the $\mathbb{R}$-algebra $\mathbb{R}[w_1, \ldots, w_s, x_1, \ldots, x_n]$ with respect to an elimination order with the x variables larger than w variables, where $\mathsf{g}_k = \mathfrak{g}_k + I$ $(1 \leq k \leq r)$.

2. Output $\mathcal{G} \cap \mathbb{R}[w_1, \ldots, w_s]$.

If the number r of the linear basis $\mathsf{g}_1, \ldots, \mathsf{g}_r$ of $\mathfrak{m}_\infty$ is not so large, as is the case in the following three examples to be discussed in the succeeding sections, it is not difficult to guess putative generators of the $\mathbb{R}$-algebra $\mathfrak{W}_\infty$ by writing out the multiplication table. If our putative generators indeed generate the $\mathbb{R}$-algebra $\mathfrak{W}_\infty$ with or without $\mathbb{R}$-algebraic redundance, we can confirm this by using Theorem 3.1, in which the ideal of relation among these putative generators is obtained as a by-product. Of course, in order to know whether there is $\mathbb{R}$-algebraic redundance among these putative generators, we must invoke Algorithm 3.2.

4 The Main Limit Diagram for the Ambrose–Palais–Singer Theorem

It is very important in the synthetic proof of the Ambrose–Palais–Singer theorem to calculate the equalizer of the following parallel arrows:

$$\mathfrak{W} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathfrak{U}, \tag{4.1}$$

where

$$\mathfrak{U} = \mathbb{R}[u_1, u_2, v_1, v_2]/\langle u_1^3, u_2^3, v_1^2, v_2^2 \rangle, \tag{4.2}$$

$$\mathfrak{W} = \mathbb{R}[x, y, z]/\langle x^3, y^2, z^2 \rangle, \tag{4.3}$$

$$\alpha(x) = u_1 u_2, \qquad \alpha(y) = v_1, \qquad \alpha(z) = v_2, \tag{4.4}$$

$$\beta(x) = u_2, \qquad \beta(y) = u_1 v_1, \qquad \beta(z) = u_1 v_2. \tag{4.5}$$

Let $\mathsf{f} \in \mathfrak{W}$, which is of the following form:

$$\begin{aligned} \mathsf{f} = a + a_1 x + a_2 y + a_3 z + a_{11} x^2 + a_{12} xy + a_{13} xz + a_{112} x^2 y \\ + a_{113} x^2 z + a_{123} xyz + a_{1123} x^2 yz. \end{aligned} \tag{4.6}$$

It is easy to see that

$$\begin{aligned} \alpha(\mathsf{f}) = a + a_1 u_1 u_2 + a_2 v_1 + a_3 v_2 + a_{11} u_1^2 u_2^2 + a_{12} u_1 u_2 v_1 + a_{13} u_1 u_2 v_2 \\ + a_{112} u_1^2 u_2^2 v_1 + a_{113} u_1^2 u_2^2 v_2 + a_{123} u_1 u_2 v_1 v_2 + a_{1123} u_1^2 u_2^2 v_1 v_2. \end{aligned} \tag{4.7}$$

On the other hand we have

$$\begin{aligned} \beta(\mathsf{f}) = a + a_1 u_2 + a_2 u_1 v_1 + a_3 u_1 v_2 + a_{11} u_2^2 + a_{12} u_1 u_2 v_1 + a_{13} u_1 u_2 v_2 \\ + a_{112} u_1 u_2^2 v_1 + a_{113} u_1 u_2^2 v_2 + a_{123} u_1^2 u_2 v_1 v_2 + a_{1123} u_1^2 u_2^2 v_1 v_2. \end{aligned} \tag{4.8}$$

Therefore f is in the equalizer of (4.1) iff the coefficients of f are previous to the following linear equations:

$$a_1 = a_2 = a_3 = 0, \tag{4.9}$$

$$a_{11} = 0, \tag{4.10}$$

$$a_{112} = a_{113} = a_{123} = 0. \tag{4.11}$$

Thus we can see that f is in the equalizer of (4.1) iff it is a linear combination of the following linearly independent polynomials:

$$1, \tag{4.12}$$

$$xy, \tag{4.13}$$

$$xz, \tag{4.14}$$

$$x^2yz. \tag{4.15}$$

It is easy to see that

$$(4.13)(4.14) = (4.15). \tag{4.16}$$

Let's compute the reduced Gröbner basis of the ideal $\langle x^3, y^2, z^2, xy - w_1, xz - w_2, x^2yz - w_3\rangle$ in the polynomial algebra $\mathbb{R}[x, y, z, w_1, w_2, w_3]$ with respect to the lexicographic order $z > y > x > w_3 > w_2 > w_1$, which goes as follows:

$$\mathcal{G} = \{w_1^2, w_2^2, w_1w_2 - w_3, xw_1w_2, x^2w_1, x^2w_2, x^3, yw_1, xy - w_1, y^2, \\ -zw_1 + yw_2, zw_2, xz - w_2, z^2\}. \tag{4.17}$$

Therefore we have

$$\mathcal{G} \cap \mathbb{R}[w_1, w_2, w_3] = \{w_1^2, w_2^2, w_1w_2 - w_3\}, \tag{4.18}$$

which reconfirms (4.16). The diagram (4.1) is now seen to be completed into the following equalizer diagram:

$$\mathfrak{W}_\infty \xrightarrow{\varphi_\infty} \mathfrak{V} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathfrak{U}, \tag{4.19}$$

where

$$\mathfrak{W}_\infty = \mathbb{R}[w_1, w_2]/\langle w_1^2, w_2^2\rangle, \tag{4.20}$$

$$\varphi_\infty(w_1) = xy \quad \text{and} \quad \varphi_\infty(w_2) = xz. \tag{4.21}$$

It remains to show that there is no $\mathbb{R}$-algebraic redundance among the generators $w_1 + I$, $w_2 + I$ of $\mathfrak{W}_\infty$, where I is the ideal $\langle w_1^2, w_2^2\rangle$. This goes as follows:

(4.22) The reduced Gröbner basis of the ideal $\langle w_1^2, w_2^2, v_1 - w_1, v_2 - w_2\rangle$ in the polynomial algebra $\mathbb{R}[w_1, w_2, v_1, v_2]$ with respect to the lexicographic order $w_2 > w_1 > v_2 > v_1$ is $\mathcal{G}_1 = \{v_1^2, v_2^2, v_1 - w_1, v_2 - w_2\}$, so that $\mathcal{G}_1 \cap \mathbb{R}[v_1, v_2] = \{v_1^2, v_2^2\}$. This means by Theorem 2.1 that $w_2 + I$ is not $\mathbb{R}$-algebraically redundant among $w_1 + I$ and $w_2 + I$.

(4.23) The reduced Gröbner basis of the ideal $\langle w_1^2, w_2^2, v_1 - w_2, v_2 - w_1\rangle$ in the polynomial algebra $\mathbb{R}[w_1, w_2, v_1, v_2]$ with respect to the lexicographic order $w_2 > w_1 > v_2 > v_1$ is $\mathcal{G}_1 = \{v_1^2, v_2^2, v_1 - w_2, v_2 - w_1\}$, so that $\mathcal{G}_1 \cap \mathbb{R}[v_1, v_2] = \{v_1^2, v_2^2\}$. This means by Theorem 2.1 that $w_1 + I$ is not $\mathbb{R}$-algebraically redundant among $w_1 + I$ and $w_2 + I$.

5 The Main Limit Diagram for the Strong Difference

We would like to find out the limit of the following diagram:

$$\begin{array}{ccc} \mathfrak{U} & \xleftarrow{\;\psi\;} & \mathfrak{W}_2 \\ {\scriptstyle\varphi}\big\uparrow & & \\ \mathfrak{W}_1 & & \end{array} \tag{5.1}$$

where

$$\mathfrak{U} = \mathbb{R}[u_1, u_2]/\langle u_1^2, u_2^2, u_1 u_2\rangle, \tag{5.2}$$

$$\mathfrak{W}_1 = \mathbb{R}[x_1, x_2]/\langle x_1^2, x_2^2\rangle, \tag{5.3}$$

$$\mathfrak{W}_2 = \mathbb{R}[y_1, y_2]/\langle y_1^2, y_2^2\rangle, \tag{5.4}$$

$$\varphi(x_1) = u_1, \qquad \varphi(x_2) = u_2, \tag{5.5}$$

$$\psi(y_1) = u_1, \qquad \psi(y_2) = u_2. \tag{5.6}$$

To this end, it suffices to calculate the equalizer of

$$\mathfrak{W}_1 \underline{\oplus} \mathfrak{W}_2 \underset{\tilde{\psi}}{\overset{\tilde{\varphi}}{\rightrightarrows}} \mathfrak{U}, \tag{5.7}$$

where

$$\mathfrak{W}_1 \underline{\oplus} \mathfrak{W}_2 = \mathbb{R}[x_1, x_2, y_1, y_2]/\langle x_1^2, x_2^2, y_1^2, y_2^2, x_1 y_1, x_1 y_2, x_2 y_1, x_2 y_2\rangle, \tag{5.8}$$

$$\tilde{\varphi}(x_1) = u_1, \qquad \tilde{\varphi}(x_2) = u_2, \qquad \tilde{\varphi}(y_1) = 0, \qquad \tilde{\varphi}(y_2) = 0, \tag{5.9}$$

$$\tilde{\psi}(x_1) = 0, \qquad \tilde{\psi}(x_2) = 0, \qquad \tilde{\psi}(y_1) = u_1, \qquad \psi(y_2) = u_2. \tag{5.10}$$

Let $\mathsf{f} \in \mathfrak{W}_1 \underline{\oplus} \mathfrak{W}_2$, which is of the following form:

$$\mathsf{f} = a + a_1^1 x_1 + a_2^1 x_2 + a_{12}^1 x_1 x_2 + a_1^2 y_1 + a_2^2 y_2 + a_{12}^2 y_1 y_2. \tag{5.11}$$

It is easy to see that

$$\tilde{\varphi}(\mathsf{f}) = a + a_1^1 u_1 + a_2^1 u_2. \tag{5.12}$$

On the other hand we have

$$\tilde{\psi}(\mathsf{f}) = a + a_1^2 u_1 + a_2^2 u_2. \tag{5.13}$$

Thus we can see that f is in the equalizer of (5.7) iff the coefficients of f are previous to the following linear equations:

$$a_1^1 = a_1^2 \quad \text{and} \quad a_2^1 = a_2^2. \tag{5.14}$$

Thus we can see that f is in the equalizer of (5.7) iff it is a linear combination of the following linearly independent polynomials:

$$1, \tag{5.15}$$

$$x_1 + y_1, \tag{5.16}$$

$$x_2 + y_2, \tag{5.17}$$

$$x_1 x_2, \tag{5.18}$$

$$y_1 y_2. \tag{5.19}$$

It is easy to see that

$$(5.16)(5.17) = (5.18) + (5.19). \tag{5.20}$$

Let's compute the reduced Gröbner basis of the ideal $\langle x_1^2, x_2^2, y_1^2, y_2^2, x_1y_1, x_2y_1, x_1y_2, x_2y_2, x_1 + y_1 - w_1, x_2 + y_2 - w_2, x_1x_2 - w_3, y_1y_2 - w_4\rangle$ in the polynomial algebra $\mathbb{R}[x_1, x_2, y_1, y_2, w_1, w_2, w_3, w_4]$ with respect to the lexicographic order $y_1 > y_2 > x_1 > x_2 > w_4 > w_3 > w_2 > w_1$, which goes as follows:

$$\begin{aligned}\mathcal{G} = \{&w_1^2, w_2^2, w_1w_3, w_2w_3, w_3^2, w_1w_2 - w_3 - w_4, -w_3 + w_1x_2, w_2x_2, w_3x_2, x_2^2,\\ &w_1x_1, -w_3 + w_2x_1, w_3x_1, -w_3 + x_1x_2, x_1^2, -w_2 + x_2 + y_2, -w_1 + x_1 + y_1\}.\end{aligned} \tag{5.21}$$

Therefore we have

$$\mathcal{G} \cap \mathbb{R}[w_1, w_2, w_3, w_4] = \{w_1^2, w_2^2, w_1w_3, w_2w_3, w_3^2, w_1w_2 - w_3 - w_4\}, \tag{5.22}$$

which reconfirms (5.20). The diagram (5.1) is now seen to be completed into the following limit diagram:

$$\begin{array}{ccc} \mathfrak{U} & \xleftarrow{\ \psi\ } & \mathfrak{W}_2 \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi_\infty} \\ \mathfrak{W}_1 & \xleftarrow{\ \psi_\infty\ } & \mathfrak{W}_\infty \end{array} \tag{5.23}$$

where

$$\mathfrak{W}_\infty = \mathbb{R}[w_1, w_2, w_3]/\langle w_1^2, w_2^2, w_3^2, w_1w_3, w_2w_3\rangle, \tag{5.24}$$

$$\varphi_\infty(w_1) = x_1, \qquad \varphi_\infty(w_2) = x_2, \qquad \varphi_\infty(w_3) = 0, \tag{5.25}$$

$$\psi_\infty(w_1) = y_1, \qquad \psi_\infty(w_2) = y_2, \qquad \psi_\infty(w_3) = y_1y_2. \tag{5.26}$$

It remains to show that there is no $\mathbb{R}$-algebraic redundance among the generators $w_1 + I$, $w_2 + I$ and $w_3 + I$ of $\mathfrak{W}_\infty$, where I is the ideal $\langle w_1^2, w_2^2, w_3^2, w_1w_3, w_2w_3\rangle$. This goes as follows:

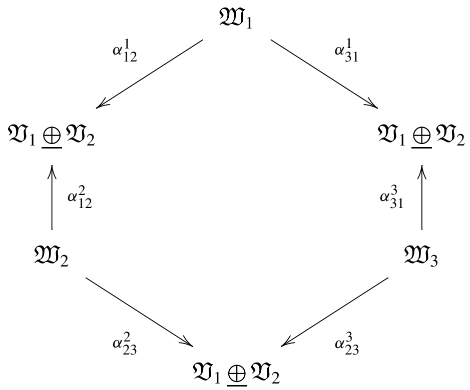
(5.27) The reduced Gröbner basis of the ideal $\langle w_1^2, w_2^2, w_1w_3, w_2w_3, w_3^2, v_1 - w_1, v_2 - w_2, v_3 - w_3\rangle$ in the polynomial algebra $\mathbb{R}[w_1, w_2, w_3, v_1, v_2, v_3]$ with respect to the lexicographic order $w_3 > w_2 > w_1 > v_3 > v_2 > v_1$ is $\mathcal{G}_1 = \{v_1^2, v_2^2, v_1v_3, v_2v_3, v_3^2, v_1 - w_1, v_2 - w_2, v_3 - w_3\}$, so that $\mathcal{G}_1 \cap \mathbb{R}[v_1, v_2, v_3] = \{v_1^2, v_2^2, v_1v_3, v_2v_3, v_3^2\}$. This means by Theorem 2.1 that $w_3 + I$ is not $\mathbb{R}$-algebraically redundant among $w_1 + I$, $w_2 + I$ and $w_3 + I$.

(5.28) The reduced Gröbner basis of the ideal $\langle w_1^2, w_2^2, w_1w_3, w_2w_3, w_3^2, v_1 - w_2, v_2 - w_3, v_3 - w_1\rangle$ in the polynomial algebra $\mathbb{R}[w_1, w_2, w_3, v_1, v_2, v_3]$ with respect to the lexicographic order $w_3 > w_2 > w_1 > v_3 > v_2 > v_1$ is $\mathcal{G}_2 = \{v_1^2, v_1v_2, v_2^2, v_2v_3, v_3^2, v_3 - w_1, v_1 - w_2, v_2 - w_3\}$, so that $\mathcal{G}_2 \cap \mathbb{R}[v_1, v_2, v_3] = \{v_1^2, v_1v_2, v_2^2, v_2v_3, v_3^2\}$. This means by Theorem 2.1 that $w_1 + I$ is not $\mathbb{R}$-algebraically redundant among $w_1 + I$, $w_2 + I$ and $w_3 + I$.

(5.29) The reduced Gröbner basis of the ideal $\langle w_1^2, w_2^2, w_1w_3, w_2w_3, w_3^2, v_1 - w_3, v_2 - w_1, v_3 - w_2\rangle$ in the polynomial algebra $\mathbb{R}[w_1, w_2, w_3, v_1, v_2, v_3]$ with respect to the lexicographic order $w_3 > w_2 > w_1 > v_3 > v_2 > v_1$ is $\mathcal{G}_3 = \{v_1^2, v_1v_2, v_2^2, v_1v_3, v_3^2, v_2 - w_1, v_3 - w_2, v_1 - w_3\}$, so that $\mathcal{G}_3 \cap \mathbb{R}[v_1, v_2, v_3] = \{v_1^2, v_1v_2, v_2^2, v_1v_3, v_3^2\}$. This means by Theorem 2.1 that $w_2 + I$ is not $\mathbb{R}$-algebraically redundant among $w_1 + I$, $w_2 + I$ and $w_3 + I$.

6 The Main Limit Diagram for the General Jacobi Identity

We would like to find out the limit of the following diagram:



, (6.1)

where

$$\mathfrak{V}_1 = \mathbb{R}[u_1, u_2, u_3]/\langle u_1^2, u_2^2, u_3^2\rangle, \tag{6.2}$$

$$\mathfrak{V}_2 = \mathbb{R}[v_1, v_2, v_3]/\langle v_1^2, v_2^2, v_3^2\rangle, \tag{6.3}$$

$$\begin{aligned}\mathfrak{W}_1 = \mathbb{R}[x_1, x_2, x_3, x_4, x_5, x_6, x_7]/\langle &x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2, x_7^2, x_2x_3, x_2x_5,\\ &x_3x_4, x_4x_5, x_2x_6, x_3x_6, x_4x_6, x_5x_6, x_1x_7, x_2x_7, x_3x_7, x_4x_7, x_5x_7, x_6x_7\rangle.\end{aligned} \tag{6.4}$$

$$\begin{aligned}\mathfrak{W}_2 = \mathbb{R}[y_1, y_2, y_3, y_4, y_5, y_6, y_7]/\langle &y_1^2, y_2^2, y_3^2, y_4^2, y_5^2, y_6^2, y_7^2, y_1y_2, y_1y_5,\\ &y_2y_4, y_4y_5, y_1y_6, y_2y_6, y_4y_6, y_5y_6, y_1y_7, y_2y_7, y_3y_7, y_4y_7, y_5y_7, y_6y_7\rangle,\end{aligned} \tag{6.5}$$

$$\mathfrak{W}_3 = \mathbb{R}[z_1, z_2, z_3, z_4, z_5, z_6, z_7]/\langle z_1^2, z_2^2, z_3^2, z_4^2, z_5^2, z_6^2, z_7^2, z_1z_2, z_1z_4, z_2z_3, z_3z_4, z_1z_6, z_2z_6, z_3z_6, z_4z_6, z_1z_7, z_2z_7, z_3z_7, z_4z_7, z_5z_7, z_6z_7\rangle, \tag{6.6}$$

$$\alpha_{12}^1(x_1) = u_1 + v_1, \quad \alpha_{12}^1(x_2) = u_2, \quad \alpha_{12}^1(x_3) = v_2, \quad \alpha_{12}^1(x_4) = u_3, \quad \alpha_{12}^1(x_5) = v_3, \quad \alpha_{12}^1(x_6) = u_2u_3, \quad \alpha_{12}^1(x_7) = 0, \tag{6.7}$$

$$\alpha_{31}^1(x_1) = u_1 + v_1, \quad \alpha_{31}^1(x_2) = v_2, \quad \alpha_{31}^1(x_3) = u_2, \quad \alpha_{31}^1(x_4) = v_3, \quad \alpha_{31}^1(x_5) = u_3, \quad \alpha_{31}^1(x_6) = u_2u_3, \quad \alpha_{31}^1(x_7) = u_1u_2u_3, \tag{6.8}$$

$$\alpha_{12}^2(y_1) = v_1, \quad \alpha_{12}^2(y_2) = u_1, \quad \alpha_{12}^2(y_3) = u_2 + v_2, \quad \alpha_{12}^2(y_4) = v_3, \quad \alpha_{12}^2(y_5) = u_3, \quad \alpha_{12}^2(y_6) = u_1u_3, \quad \alpha_{12}^2(y_7) = u_1u_2u_3, \tag{6.9}$$

$$\alpha_{23}^2(y_1) = u_1, \quad \alpha_{23}^2(y_2) = v_1, \quad \alpha_{23}^2(y_3) = u_2 + v_2, \quad \alpha_{23}^2(y_4) = u_3, \quad \alpha_{23}^2(y_5) = v_3, \quad \alpha_{23}^2(y_6) = u_1u_3, \quad \alpha_{23}^2(y_7) = 0, \tag{6.10}$$

$$\alpha_{23}^3(z_1) = v_1, \quad \alpha_{23}^3(z_2) = u_1, \quad \alpha_{23}^3(z_3) = v_2, \quad \alpha_{23}^3(z_4) = u_2, \quad \alpha_{23}^3(z_5) = u_3 + v_3, \quad \alpha_{23}^3(z_6) = u_1u_2, \quad \alpha_{23}^3(z_7) = u_1u_2u_3, \tag{6.11}$$

$$\alpha_{31}^3(z_1) = u_1, \quad \alpha_{31}^3(z_2) = v_1, \quad \alpha_{31}^3(z_3) = u_2, \quad \alpha_{31}^3(z_4) = v_2, \quad \alpha_{31}^3(z_5) = u_3 + v_3, \quad \alpha_{31}^3(z_6) = u_1u_2, \quad \alpha_{31}^3(z_7) = 0. \tag{6.12}$$

To this end, it suffices to compute the equalizer of

$$\mathfrak{W}_1 \underline{\oplus} \mathfrak{W}_2 \underline{\oplus} \mathfrak{W}_3 \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \mathfrak{V}_1 \underline{\oplus} \mathfrak{V}_2 \underline{\oplus} \mathfrak{V}_1 \underline{\oplus} \mathfrak{V}_2 \underline{\oplus} \mathfrak{V}_1 \underline{\oplus} \mathfrak{V}_2, \tag{6.13}$$

where, if $\pi_i : \mathfrak{W}_1 \underline{\oplus} \mathfrak{W}_2 \underline{\oplus} \mathfrak{W}_3 \to \mathfrak{W}_i$ $(1 \le i \le 3)$ denotes the canonical projection and $\rho_i : \mathfrak{V}_1 \underline{\oplus} \mathfrak{V}_2 \underline{\oplus} \mathfrak{V}_1 \underline{\oplus} \mathfrak{V}_2 \underline{\oplus} \mathfrak{V}_1 \underline{\oplus} \mathfrak{V}_2 \to \mathfrak{V}_1 \underline{\oplus} \mathfrak{V}_2$ denotes the canonical projection to the ith $\mathfrak{V}_1 \underline{\oplus} \mathfrak{V}_2$ in $\mathfrak{V}_1 \underline{\oplus} \mathfrak{V}_2 \underline{\oplus} \mathfrak{V}_1 \underline{\oplus} \mathfrak{V}_2 \underline{\oplus} \mathfrak{V}_1 \underline{\oplus} \mathfrak{V}_2$ $(1 \le i \le 3)$, then φ and ψ are characterized as follows:

$$\rho_1 \circ \varphi = \alpha_{12}^1 \circ \pi_1, \qquad \rho_2 \circ \varphi = \alpha_{23}^2 \circ \pi_2, \quad \text{and} \quad \rho_3 \circ \varphi = \alpha_{31}^3 \circ \pi_3, \tag{6.14}$$

$$\rho_1 \circ \psi = \alpha_{31}^1 \circ \pi_1, \qquad \rho_2 \circ \psi = \alpha_{12}^2 \circ \pi_2, \quad \text{and} \quad \rho_3 \circ \psi = \alpha_{23}^3 \circ \pi_3. \tag{6.15}$$

Let $\mathsf{f} \in \mathfrak{W}_1 \underline{\oplus} \mathfrak{W}_2 \underline{\oplus} \mathfrak{W}_3$, which is of the following form:

$$\begin{aligned}\mathsf{f} = {} & a + a_1^1x_1 + a_2^1x_2 + a_3^1x_3 + a_4^1x_4 + a_5^1x_5 + a_6^1x_6 + a_7^1x_7 + a_{12}^1x_1x_2 + a_{13}^1x_1x_3 \\ & + a_{14}^1x_1x_4 + a_{15}^1x_1x_5 + a_{16}^1x_1x_6 + a_{24}^1x_2x_4 + a_{35}^1x_3x_5 + a_{124}^1x_1x_2x_4 + a_{135}^1x_1x_3x_5 \\ & + a_1^2y_1 + a_2^2y_2 + a_3^2y_3 + a_4^2y_4 + a_5^2y_5 + a_6^2y_6 + a_7^2y_7 + a_{13}^2y_1y_3 + a_{14}^2y_1y_4 \\ & + a_{23}^2y_2y_3 + a_{25}^2y_2y_5 + a_{34}^2y_3y_4 + a_{35}^2y_3y_5 + a_{36}^2y_3y_6 + a_{134}^2y_1y_3y_4 + a_{235}^2y_2y_3y_5 \\ & + a_1^3z_1 + a_2^3z_2 + a_3^3z_3 + a_4^3z_4 + a_5^3z_5 + a_6^3z_6 + a_7^3z_7 + a_{13}^3z_1z_3 + a_{15}^3z_1z_5 + a_{24}^3z_2z_4 \\ & + a_{25}^3z_2z_5 + a_{35}^3z_3z_5 + a_{45}^3z_4z_5 + a_{56}^3z_5z_6 + a_{135}^3z_1z_3z_5 + a_{245}^3z_2z_4z_5.\end{aligned} \tag{6.16}$$

Let $\mathfrak{V}_1 \underline{\oplus} \mathfrak{V}_2 \underline{\oplus} \mathfrak{V}_1 \underline{\oplus} \mathfrak{V}_2 \underline{\oplus} \mathfrak{V}_1 \underline{\oplus} \mathfrak{V}_2 = \mathbb{R}[u_1, u_2, u_3, v_1, v_2, v_3, u_1', u_2', u_3', v_1', v_2', v_3', u_1'', u_2'', u_3'', v_1'', v_2'', v_3'']\,/\,(u_1^2, u_2^2, u_3^2, v_1^2, v_2^2, v_3^2, u_1'^2, u_2'^2, u_3'^2, v_1'^2, v_2'^2, v_3'^2, u_1''^2, u_2''^2, u_3''^2, v_1''^2, v_2''^2, v_3''^2,$

$u_1v_1, u_1v_2, u_1v_3, u_1u_1', u_1u_2', u_1u_3', u_1v_1', u_1v_2', u_1v_3', u_1u_1'', u_1u_2'', u_1u_3'', u_1v_1'', u_1v_2'', u_1v_3''$,
$u_2v_1, u_2v_2, u_2v_3, u_2u_1', u_2u_2', u_2u_3', u_2v_1', u_2v_2', u_2v_3', u_2u_1'', u_2u_2'', u_2u_3'', u_2v_1'', u_2v_2'', u_2v_3''$,
$u_3v_1, u_3v_2, u_3v_3, u_3u_1', u_3u_2', u_3u_3', u_3v_1', u_3v_2', u_3v_3', u_3u_1'', u_3u_2'', u_3u_3'', u_3v_1'', u_3v_2'', u_3v_3''$,
$v_1u_1', v_1u_2', v_1u_3', v_1v_1', v_1v_2', v_1v_3', v_1u_1'', v_1u_2'', v_1u_3'', v_1v_1'', v_1v_2'', v_1v_3'', v_2u_1', v_2u_2', v_2u_3', v_2v_1'$,
$v_2v_2', v_2v_3', v_2u_1'', v_2u_2'', v_2u_3'', v_2v_1'', v_2v_2'', v_2v_3'', v_3u_1', v_3u_2', v_3u_3', v_3v_1', v_3v_2', v_3v_3', v_3u_1'', v_3u_2''$,
$v_3u_3'', v_3v_1'', v_3v_2'', v_3v_3'', u_1'v_1', u_1'v_2', u_1'v_3', u_1'u_1'', u_1'u_2'', u_1'u_3'', u_1'v_1'', u_1'v_2'', u_1'v_3'', u_2'v_1', u_2'v_2'$,
$u_2'v_3', u_2'u_1'', u_2'u_2'', u_2'u_3'', u_2'v_1'', u_2'v_2'', u_2'v_3'', u_3'v_1', u_3'v_2', u_3'v_3', u_3'u_1'', u_3'u_2'', u_3'u_3'', u_3'v_1'', u_3'v_2''$,
$u_3'v_3'', v_1'u_1'', v_1'u_2'', v_1'u_3'', v_1'v_1'', v_1'v_2'', v_1'v_3'', v_2'u_1'', v_2'u_2'', v_2'u_3'', v_2'v_1'', v_2'v_2'', v_2'v_3'', v_3'u_1'', v_3'u_2''$,
$v_3'u_3'', v_3'v_1'', v_3'v_2'', v_3'v_3'', u_1''v_1'', u_1''v_2'', u_1''v_3'', u_2''v_1'', u_2''v_2'', u_2''v_3'', u_3''v_1'', u_3''v_2'', u_3''v_3'')$.

It is easy to see that

$$\begin{aligned}
\varphi(\mathsf{f}) &= a + a_1^1(u_1+v_1) + a_2^1u_2 + a_3^1v_2 + a_4^1u_3 + a_5^1v_3a_6^1u_2u_3 + a_{12}^1(u_1+v_1)u_2\\
&\quad + a_{13}^1(u_1+v_1)v_2 + a_{14}^1(u_1+v_1)u_3 + a_{15}^1(u_1+v_1)v_3 + a_{16}^1(u_1+v_1)u_2u_3\\
&\quad + a_{24}^1u_2u_3 + a_{35}^1v_2v_3 + a_{124}^1(u_1+v_1)u_2u_3 + a_{135}^1(u_1+v_1)v_2v_3 + a_1^2u_1'\\
&\quad + a_2^2v_1' + a_3^2(u_2'+v_2') + a_4^2u_3' + a_5^2v_3' + a_6^2u_1'u_3' + a_{13}^2u_1'(u_2'+v_2')\\
&\quad + a_{14}^2u_1'u_3' + a_{23}^2v_1'(u_2'+v_2')_1 + a_{25}^2v_1'v_3' + a_{34}^2(u_2'+v_2')u_3' + a_{35}^2(u_2'+v_2')v_3'\\
&\quad + a_{36}^2(u_2'+v_2')u_1'u_3' + a_{134}^2u_1'(u_2'+v_2')u_3' + a_{235}^2v_1'(u_2'+v_2')v_3' + a_1^3u_1''\\
&\quad + a_2^3v_1'' + a_3^3u_2'' + a_4^3v_2'' + a_5^3(u_3''+v_3'') + a_6^3u_1''u_2'' + a_{13}^3u_1''u_2'' + a_{15}^3u_1''(u_3''+v_3'')\\
&\quad + a_{24}^3v_1''v_2'' + a_{25}^3v_1''(u_3''+v_3'') + a_{35}^3u_2''(u_3''+v_3'') + a_{45}^3v_2''(u_3''+v_3'')\\
&\quad + a_{56}^3(u_3''+v_3'')u_1''u_2'' + a_{135}^3u_1''u_2''(u_3''+v_3'') + a_{245}^3v_1''v_2''(u_3''+v_3'')\\
&= a + a_1^1u_1 + a_1^1v_1 + a_2^1u_2 + a_3^1v_2 + a_4^1u_3 + a_5^1v_3 + a_{12}^1u_1u_2 + a_{13}^1v_1v_2\\
&\quad + a_{14}^1u_1u_3 + a_{15}^1v_1v_3 + (a_{24}^1 + a_6^1)u_2u_3 + a_{35}^1v_2v_3 + (a_{16}^1 + a_{124}^1)u_1u_2u_3\\
&\quad + a_{135}^1v_1v_2v_3 + a_1^2u_1' + a_2^2v_1' + a_3^2u_2' + a_3^2v_2' + a_4^2u_3' + a_5^2v_3' + a_{13}^2u_1'u_2' + a_{23}^2v_1'v_2'\\
&\quad + (a_6^2 + a_{14}^2)u_1'u_3' + a_{25}^2v_1'v_3' + a_{34}^2u_2'u_3' + a_{35}^2v_2'v_3' + (a_{36}^2 + a_{134}^2)u_1'u_2'u_3'\\
&\quad + a_{235}^2v_1'v_2'v_3' + a_1^3u_1'' + a_2^3v_1'' + a_3^3u_2'' + a_4^3v_2'' + a_5^3u_3'' + a_5^3v_3'' + (a_{13}^3 + a_6^3)u_1''u_2''\\
&\quad + a_{24}^3v_1''v_2'' + a_{15}^3u_1''u_3'' + a_{25}^3v_1''v_3'' + a_{35}^3u_2''u_3'' + a_{45}^3v_2''v_3'' + (a_{56}^3 + a_{135}^3)u_1''u_2''u_3''\\
&\quad + a_{245}^3v_1''v_2''v_3''.
\end{aligned}\tag{6.17}$$

On the other hand we have

$$\begin{aligned}
\psi(\mathsf{f}) &= a + a_1^1(u_1+v_1) + a_2^1v_2 + a_3^1u_2 + a_4^1v_3 + a_5^1u_3 + a_6^1u_2u_3 + a_7^1u_1u_2u_3\\
&\quad + a_{12}^1(u_1+v_1)v_2 + a_{13}^1(u_1+v_1)u_2 + a_{14}^1(u_1+v_1)v_3 + a_{15}^1(u_1+v_1)u_3\\
&\quad + a_{16}^1(u_1+v_1)u_2u_3 + a_{24}^1v_2v_3 + a_{35}^1u_2u_3 + a_{124}^1(u_1+v_1)v_2v_3\\
&\quad + a_{135}^1(u_1+v_1)u_2u_3 + a_1^2v_1' + a_2^2u_1' + a_3^2(u_2'+v_2') + a_4^2v_3' + a_5^2u_3'\\
&\quad + a_6^2u_1'u_3' + a_7^2u_1'u_2'u_3' + a_{13}^2v_1'(u_2'+v_2') + a_{14}^2v_1'v_3' + a_{23}^2u_1'(u_2'+v_2')\\
&\quad + a_{25}^2u_1'u_3' + a_{34}^2(u_2'+v_2')v_3' + a_{35}^2(u_2'+v_2')u_3' + a_{36}^2(u_2'+v_2')u_1'u_3'\\
&\quad + a_{134}^2v_1'(u_2'+v_2')v_3' + a_{235}^2u_1'(u_2'+v_2')u_3' + a_1^3v_1'' + a_2^3u_1'' + a_3^3v_2'' + a_4^3u_2''\\
&\quad + a_5^3(u_3''+v_3'') + a_6^3u_1''u_2'' + a_7^3u_1''u_2''u_3'' + a_{13}^3v_1''v_2'' + a_{15}^3v_1''(u_3''+v_3'')
\end{aligned}$$

$$
\begin{aligned}
&+ a^3_{24}u''_1u''_2 + a^3_{25}u''_1(u''_3 + v''_3) + a^3_{35}v''_2(u''_3 + v''_3) + a^3_{45}u''_2(u''_3 + v''_3) \\
&+ a^3_{56}(u''_3 + v''_3)u''_1u''_2 + a^3_{135}v''_1v''_2(u''_3 + v''_3) + a^3_{245}u''_1u''_2(u''_3 + v''_3) \\
= {}& a + a^1_1u_1 + a^1_1v_1 + a^1_3u_2 + a^1_2v_2 + a^1_5u_3 + a^1_4v_3 + a^1_{13}u_1u_2 + a^1_{12}v_1v_2 \\
&+ a^1_{15}u_1u_3 + a^1_{14}v_1v_3 + (a^1_6 + a^1_{35})u_2u_3 + a^1_{24}v_2v_3 + (a^1_7 + a^1_{16} + a^1_{135})u_1u_2u_3 \\
&+ a^1_{124}v_1v_2v_3 + a^2_2u'_1 + a^2_1v'_1 + a^2_3u'_2 + a^2_3v'_2 + a^2_5u'_3 + a^2_4v'_3a^2_{23}u'_1u'_2 + a^2_{13}v'_1v'_2 \\
&+ (a^2_6 + a^2_{25})u'_1u'_2 + a^2_{14}v'_1v'_3 + a^2_{35}u'_2u'_3 + a^2_{34}v'_2v'_3 + (a^2_7 + a^2_{36} + a^2_{235})u'_1u'_2u'_3 \\
&+ a^2_{134}v'_1v'_2v'_3 + a^3_2u''_1 + a^3_1v''_1 + a^3_4u''_2 + a^3_3v''_2 + a^3_5u''_3 + a^3_5v''_3 + (a^3_6 + a^3_{24})u''_1u''_2 \\
&+ a^3_{13}v''_1v''_2 + a^3_{25}u''_1u''_3 + a^3_{15}v''_1v''_3 + a^3_{45}u''_2u''_3 + a^3_{35}v''_2v''_3 \\
&+ (a^3_7 + a^3_{56} + a^3_{245})u''_1u''_2u''_3 + a^3_{135}v''_1v''_2v''_3.
\end{aligned}
\tag{6.18}
$$

Therefore f is in the equalizer of (6.13) iff the coefficients of facquiesce in the following linear equations:

$$a^1_1 = a^2_1 = a^2_2 = a^3_1 = a^3_2, \tag{6.19}$$

$$a^1_2 = a^1_3 = a^2_3 = a^3_3 = a^3_4, \tag{6.20}$$

$$a^1_4 = a^1_5 = a^2_4 = a^2_5 = a^3_5, \tag{6.21}$$

$$a^1_{24} = a^1_{35} = a^2_{34} = a^3_{45}, \tag{6.22}$$

$$a^1_{14} = a^2_{14} = a^2_{25} = a^3_{15}, \tag{6.23}$$

$$a^1_{12} = a^2_{23} = a^3_{13} = a^3_{24}, \tag{6.24}$$

$$a^1_{13} = a^2_{13}, \tag{6.25}$$

$$a^1_{14} = a^3_{25}, \tag{6.26}$$

$$a^2_{35} = a^3_{35}, \tag{6.27}$$

$$a^1_6 = a^3_{35} - a^1_{35} = a^2_{35} - a^1_{24}, \tag{6.28}$$

$$a^2_6 = a^3_{25} - a^2_{14} = a^1_{14} - a^2_{25}, \tag{6.29}$$

$$a^3_6 = a^1_{13} - a^3_{13} = a^2_{13} - a^3_{24}, \tag{6.30}$$

$$a^1_{124} + a^1_{16} = a^2_{235} + a^2_{36} + a^2_7, \tag{6.31}$$

$$a^2_{134} + a^2_{36} = a^3_{245} + a^3_{56} + a^3_7, \tag{6.32}$$

$$a^3_{135} + a^3_{56} = a^1_{135} + a^1_{16} + a^1_7, \tag{6.33}$$

$$a^1_{135} = a^2_{134}, \tag{6.34}$$

$$a^2_{235} = a^3_{135}, \tag{6.35}$$

$$a^3_{245} = a^1_{124}. \tag{6.36}$$

Thus we see that f is in the equalizer of (4.16) iff it is a linear combination of the following linearly independent polynomials.

$$1, \tag{6.37}$$

$$x_1 + y_1 + y_2 + z_1 + z_2, \tag{6.38}$$

$$x_2 + x_3 + y_3 + z_3 + z_4, \tag{6.39}$$

$$x_4 + x_5 + y_4 + y_5 + z_5, \tag{6.40}$$

$$x_6 + y_3 y_5 + z_3 z_5, \tag{6.41}$$

$$y_6 + x_1 x_4 + z_2 z_5, \tag{6.42}$$

$$z_6 + x_1 x_3 + y_1 y_3, \tag{6.43}$$

$$x_6 - x_2 x_4 - x_3 x_5 - y_3 y_4 - z_4 z_5, \tag{6.44}$$

$$y_6 - x_1 x_5 - y_1 y_4 - y_2 y_5 - z_1 z_5, \tag{6.45}$$

$$z_6 - x_1 x_2 - y_2 y_3 - z_1 z_3 - z_2 z_4, \tag{6.46}$$

$$x_7 - y_7 - x_1 x_6, \tag{6.47}$$

$$y_7 - z_7 - y_3 y_6, \tag{6.48}$$

$$z_7 - x_7 - z_5 z_6, \tag{6.49}$$

$$x_7 - y_7 + y_2 y_3 y_5 + z_1 z_3 z_5, \tag{6.50}$$

$$y_7 - z_7 + x_1 x_2 x_4 + z_2 z_4 z_5, \tag{6.51}$$

$$z_7 - x_7 + x_1 x_3 x_5 + y_1 y_3 y_4. \tag{6.52}$$

By writing out the multiplication table for (6.37–6.52), it is easy to see that

$$(6.38)(6.39) = (6.43) - (6.46), \tag{6.53}$$

$$(6.38)(6.40) = (6.42) - (6.45), \tag{6.54}$$

$$(6.39)(6.40) = (6.41) - (6.44), \tag{6.55}$$

$$(6.38)(6.39)(6.40) = (6.50) + (6.51) + (6.52), \tag{6.56}$$

$$(6.38)(6.41) = (6.50) - (6.47), \tag{6.57}$$

$$(6.39)(6.42) = (6.51) - (6.48), \tag{6.58}$$

$$(6.40)(6.43) = (6.52) - (6.49). \tag{6.59}$$

Let's compute the reduced Gröbner basis of the ideal $\langle x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2, x_7^2, x_2x_3, x_2x_5, x_3x_4, x_4x_5, x_2x_6, x_3x_6, x_4x_6, x_5x_6, x_1x_7, x_2x_7, x_3x_7, x_4x_7, x_5x_7, x_6x_7, y_1^2, y_2^2, y_3^2, y_4^2, y_5^2, y_6^2, y_7^2, y_1y_2, y_1y_5, y_2y_4, y_4y_5, y_1y_6, y_2y_6, y_4y_6, y_5y_6, y_1y_7, y_2y_7, y_3y_7, y_4y_7, y_5y_7, y_6y_7, z_1^2, z_2^2, z_3^2, z_4^2, z_5^2, z_6^2, z_7^2, z_1z_2, z_1z_4, z_2z_3, z_3z_4, z_1z_6, z_2z_6, z_3z_6, z_4z_6, z_1z_7, z_2z_7, z_3z_7, z_4z_7, z_5z_7, z_6z_7, x_1y_1, x_1y_2, x_1y_3, x_1y_4, x_1y_5, x_1y_6, x_1y_7, x_2y_1, x_2y_2, x_2y_3, x_2y_4, x_2y_5, x_2y_6, x_2y_7, x_3y_1, x_3y_2, x_3y_3, x_3y_4, x_3y_5, x_3y_6, x_3y_7, x_4y_1, x_4y_2, x_4y_3, x_4y_4, x_4y_5, x_4y_6, x_4y_7, x_5y_1, x_5y_2, x_5y_3, x_5y_4, x_5y_5, x_5y_6, x_5y_7, x_6y_1, x_6y_2, x_6y_3, x_6y_4, x_6y_5, x_6y_6, x_6y_7, x_7y_1, x_7y_2, x_7y_3, x_7y_4, x_7y_5, x_7y_6, x_7y_7, y_1z_1, y_1z_2, y_1z_3, y_1z_4, y_1z_5, y_1z_6, y_1z_7, y_2z_1, y_2z_2, y_2z_3, y_2z_4, y_2z_5, y_2z_6, y_2z_7, y_3z_1, y_3z_2, y_3z_3, y_3z_4, y_3z_5, y_3z_6, y_3z_7, y_4z_1, y_4z_2, y_4z_3, y_4z_4, y_4z_5, y_4z_6, y_4z_7, y_5z_1, y_5z_2, y_5z_3, y_5z_4, y_5z_5, y_5z_6, y_5z_7, y_6z_1, y_6z_2, y_6z_3, y_6z_4, y_6z_5, y_6z_6, y_6z_7, y_7z_1, y_7z_2, y_7z_3, y_7z_4, y_7z_5, y_7z_6, y_7z_7, x_1z_1, x_1z_2, x_1z_3, x_1z_4, x_1z_5, x_1z_6, x_1z_7, x_2z_1, x_2z_2, x_2z_3, x_2z_4, x_2z_5, x_2z_6, x_2z_7,$

$x_3z_1, x_3z_2, x_3z_3, x_3z_4, x_3z_5, x_3z_6, x_3z_7, x_4z_1, x_4z_2, x_4z_3, x_4z_4, x_4z_5, x_4z_6, x_4z_7, x_5z_1, x_5z_2, x_5z_3, x_5z_4, x_5z_5, x_5z_6, x_5z_7, x_6z_1, x_6z_2, x_6z_3, x_6z_4, x_6z_5, x_6z_6, x_6z_7, x_7z_1, x_7z_2, x_7z_3, x_7z_4, x_7z_5, x_7z_6, x_7z_7, x_1+y_1+y_2+z_1+z_2-w_1, x_2+x_3+y_3+z_3+z_4-w_2, x_4+x_5+y_4+y_5+z_5-w_3, x_6+y_3y_5+z_3z_5-w_4, y_6+x_1x_4+z_2z_5-w_5, z_6+x_1x_3+y_1y_3-w_6, x_7-y_7-x_1x_6-w_7, y_7-z_7-y_3y_6-w_8, z_7-x_7-z_5z_6-w_9, x_6-x_2x_4-x_3x_5-y_3y_4-z_4z_5-w_{10}, y_6-x_1x_5-y_1y_4-y_2y_5-z_1z_5-w_{11}, z_6-x_1x_2-y_2y_3-z_1z_3-z_2z_4-w_{12}, x_7-y_7+y_2y_3y_5+z_1z_3z_5-w_{13}, y_7-z_7+x_1x_2x_4+z_2z_4z_5-w_{14}, z_7-x_7+x_1x_3x_5+y_1y_3y_4-w_{15}\rangle$ in the polynomial algebra $\mathbb{R}[x_1, x_2, x_3, x_4, x_5, x_6, x_7, y_1, y_2, y_3, y_4, y_5, y_6, y_7, z_1, z_2, z_3, z_4, z_5, z_6, z_7, w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, w_{10}, w_{11}, w_{12}, w_{13}, w_{14}, w_{15}]$ with respect to the lexicographic order $z_1 > z_2 > z_3 > z_4 > z_5 > z_6 > z_7 > y_1 > y_2 > y_3 > y_4 > y_5 > y_6 > y_7 > x_1 > x_2 > x_3 > x_4 > x_5 > x_6 > x_7 > w_{15} > w_{14} > w_{13} > w_{12} > w_{11} > w_{10} > w_9 > w_8 > w_7 > w_6 > w_5 > w_4 > w_3 > w_2 > w_1$, which goes as follows:

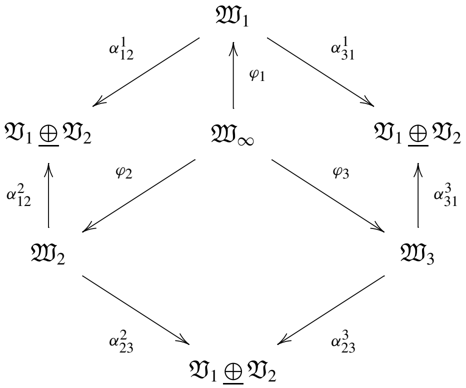
$$\begin{aligned}
\mathcal{G} = \{&w_1^2, w_2^2, w_3^2, w_2w_4, w_3w_4, w_4^2, w_1w_5, w_3w_5, w_4w_5, w_5^2, w_1w_6, w_2w_6, w_4w_6,\\
&w_5w_6, w_6^2, w_1w_7, w_2w_7, w_3w_7, w_4w_7, w_5w_7, w_6w_7, w_7^2, w_1w_8, w_2w_8, w_3w_8, w_4w_8,\\
&w_5w_8, w_6w_8, w_7w_8, w_8^2, -w_1w_2w_3+w_1w_4+w_2w_5+w_3w_6+w_7+w_8+w_9,\\
&w_2w_3-w_4+w_{10}, -w_1w_3+w_5-w_{11}, w_1w_2-w_6+w_{12}, -w_1w_4-w_7+w_{13},\\
&w_2w_5+w_8-w_{14}, w_1w_2w_3-w_1w_4-w_2w_5-w_7-w_8-w_{15}, w_1x_7, w_2x_7, w_3x_7,\\
&w_4x_7, w_5x_7, w_6x_7, w_7x_7, w_8x_7, x_7^2, w_2x_6, w_3x_6, w_4x_6, w_5x_6, w_6x_6, w_7x_6, w_8x_6,\\
&x_6x_7, x_6^2, w_3x_5, w_4x_5, w_5x_5, -w_1w_2x_5+w_6x_5, w_7x_5, w_8x_5, x_5x_7, x_5x_6, x_5^2, w_3x_4,\\
&w_4x_4, w_5x_4, w_6x_4, w_7x_4, w_8x_4, x_4x_7, x_4x_6, x_4x_5, x_4^2, w_2x_3, w_3x_3-w_2x_5, w_4x_3,\\
&w_5x_3, w_6x_3, w_7x_3, w_8x_3, x_3x_7, x_3x_6, w_2x_5-x_3x_5, x_3x_4, x_3^2, w_2x_2, w_3x_2-w_2x_4,\\
&w_4x_2, w_5x_2-w_1w_2x_4, w_6x_2, w_7x_2, w_8x_2, x_2x_7, x_2x_6, x_2x_5, w_2x_4-x_2x_4, x_2x_3,\\
&x_2^2, w_1x_1, w_2x_1-w_1x_2-w_1x_3, w_3x_1-w_1x_4-w_1x_5, w_4x_1-w_1x_6, w_5x_1, w_6x_1,\\
&w_7x_1, w_8x_1, x_1x_7, w_1x_6-x_1x_6, w_1x_5-x_1x_5, -w_1x_4+x_1x_4, w_1x_3-x_1x_3,\\
&-w_1x_2+x_1x_2, x_1^2, w_7+w_1x_6-x_7+y_7, w_1y_6, w_3y_6, w_4y_6, w_5y_6, w_6y_6,\\
&w_7y_6, w_8y_6, x_7y_6, x_6y_6, x_5y_6, x_4y_6, x_3y_6, x_2y_6, x_1y_6, y_6^2, w_3y_5, w_4y_5, w_5y_5,\\
&w_6y_5, w_7y_5, w_8y_5, x_7y_5, x_6y_5, x_5y_5, x_4y_5, x_3y_5, x_2y_5, x_1y_5, y_5y_6, y_5^2,\\
&-w_1w_2w_3+w_1w_4+w_2w_5+w_1w_2x_5-w_1x_6+w_1w_2y_4-w_2y_6, w_3y_4,\\
&w_4y_4, w_5y_4, -w_1w_2w_3+w_1w_4+w_2w_5+w_1w_2x_5-w_1x_6+w_6y_4-w_2y_6,\\
&w_7y_4, w_8y_4, x_7y_4, x_6y_4, x_5y_4, x_4y_4, x_3y_4, x_2y_4, x_1y_4, y_4y_6, y_4y_5, y_4^2, w_2y_3,\\
&w_3y_3-w_2y_4-w_2y_5, w_4y_3, w_5y_3-w_2y_6, w_6y_3, w_7y_3, w_8y_3, x_7y_3, x_6y_3, x_5y_3,\\
&x_4y_3, x_3y_3, x_2y_3, x_1y_3, w_2y_6-y_3y_6, w_2y_5-y_3y_5, -w_2y_4+y_3y_4, y_3^2, w_1y_2,\\
&w_3y_2-w_1y_5, -w_4y_2+w_1w_2y_5, w_5y_2, w_6y_2, w_7y_2, w_8y_2, x_7y_2, x_6y_2, x_5y_2,\\
&x_4y_2, x_3y_2, x_2y_2, x_1y_2, y_2y_6, w_1y_5-y_2y_5, y_2y_4, w_2y_2-y_2y_3, y_2^2, w_1y_1,\\
&-w_2y_1-w_2y_2+w_1y_3, w_3y_1-w_1y_4, w_4y_1, w_5y_1, w_6y_1, w_7y_1, w_8y_1, x_7y_1,\\
&x_6y_1, x_5y_1, x_4y_1, x_3y_1, x_2y_1, x_1y_1, y_1y_6, y_1y_5, w_1y_4-y_1y_4, w_2y_2-w_1y_3+y_1y_3,
\end{aligned}$$

$$
\begin{aligned}
&y_1y_2,\, y_1^2,\, -w_7 - w_8 - w_1x_6 + x_7 - w_2y_6 - z_7,\, w_6 - w_1x_3 + w_2y_2 - w_1y_3 - z_6,\\
&-w_3 + x_4 + x_5 + y_4 + y_5 + z_5,\, w_2z_4,\\
&w_2w_3 - w_4 - w_2x_4 - w_2x_5 + x_6 - w_2y_4 - w_3z_4,\, w_4z_4,\\
&w_2w_5 - w_1w_2x_4 - w_2y_6 - w_5z_4,\, w_6z_4,\, w_7z_4,\, w_8z_4,\, x_7z_4,\, x_6z_4,\, x_5z_4,\, x_4z_4,\, x_3z_4,\\
&x_2z_4,\, x_1z_4,\, y_6z_4,\, y_5z_4,\, y_4z_4,\, y_3z_4,\, y_2z_4,\, y_1z_4,\, z_4^2,\, -w_2 + x_2 + x_3 + y_3 + z_3 + z_4,\\
&w_1z_2,\, -w_2z_2 + w_1z_4,\, -w_5 + w_1x_4 + y_6 + w_3z_2,\, w_4z_2,\, w_5z_2,\, w_6z_2,\, w_7z_2,\, w_8z_2,\\
&x_7z_2,\, x_6z_2,\, x_5z_2,\, x_4z_2,\, x_3z_2,\, x_2z_2,\, x_1z_2,\, y_6z_2,\, y_5z_2,\, y_4z_2,\, y_3z_2,\, y_2z_2,\, y_1z_2,\\
&w_1z_4 - z_2z_4,\, z_2^2,\, -w_1 + x_1 + y_1 + y_2 + z_1 + z_2\}.
\end{aligned}
\tag{6.60}
$$

Therefore we have

$$
\begin{aligned}
&\mathcal{G} \cap \mathbb{R}[w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, w_{10}, w_{11}, w_{12}, w_{13}, w_{14}, w_{15}]\\
&\quad = \{w_1^2, w_2^2, w_3^2, w_2w_4, w_3w_4, w_4^2, w_1w_5, w_3w_5, w_4w_5, w_5^2, w_1w_6, w_2w_6, w_4w_6, w_5w_6,\\
&\qquad w_6^2, w_1w_7, w_2w_7, w_3w_7, w_4w_7, w_5w_7, w_6w_7, w_7^2, w_1w_8, w_2w_8, w_3w_8, w_4w_8,\\
&\qquad w_5w_8, w_6w_8, w_7w_8, w_8^2, -w_1w_2w_3 + w_1w_4 + w_2w_5 + w_3w_6 + w_7 + w_8 + w_9,\\
&\qquad w_2w_3 - w_4 + w_{10}, -w_1w_3 + w_5 - w_{11}, w_1w_2 - w_6 + w_{12}, -w_1w_4 - w_7 + w_{13},\\
&\qquad w_2w_5 + w_8 - w_{14}, w_1w_2w_3 - w_1w_4 - w_2w_5 - w_7 - w_8 - w_{15}\},
\end{aligned}
\tag{6.61}
$$

which shows clearly that the variables $w_9, w_{10}, w_{11}, w_{12}, w_{13}, w_{14}, w_{15}$ are $\mathbb{R}$-algebraically redundant. The diagram (6.1) is now seen to be completed into the following limit diagram:



(6.62)

where

$$
\begin{aligned}
\mathfrak{W}_\infty = \mathbb{R}[&w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8]/\langle w_1^2, w_2^2, w_3^2, w_4^2, w_5^2, w_6^2, w_7^2, w_8^2, w_1w_5,\\
&w_1w_6, w_2w_4, w_2w_6, w_3w_4, w_3w_5, w_4w_5, w_4w_6, w_5w_6, w_1w_7, w_1w_8, w_2w_7,\\
&w_2w_8, w_3w_7, w_3w_8, w_4w_7, w_4w_8, w_5w_7, w_5w_8, w_6w_7, w_6w_8, w_7w_8\rangle,
\end{aligned}
\tag{6.63}
$$

$$\varphi_1(w_1)=x_1,\quad \varphi_1(w_2)=x_2+x_3,\quad \varphi_1(w_3)=x_4+x_5,\quad \varphi_1(w_4)=x_6,$$
$$\varphi_1(w_5)=x_1x_4,\quad \varphi_1(w_6)=x_1x_3,\quad \varphi_1(w_7)=x_7-x_1x_6,\quad \varphi_1(w_8)=0, \tag{6.64}$$
$$\varphi_2(w_1)=y_1+y_2,\quad \varphi_2(w_2)=y_3,\quad \varphi_2(w_3)=y_4+y_5,\quad \varphi_2(w_4)=y_3y_5,$$
$$\varphi_2(w_5)=y_6,\quad \varphi_2(w_6)=y_1y_3,\quad \varphi_2(w_7)=-y_7,\quad \varphi_2(w_8)=y_7-y_3y_6, \tag{6.65}$$
$$\varphi_3(w_1)=z_1+z_2,\quad \varphi_3(w_2)=z_3+z_4,\quad \varphi_3(w_3)=z_5,\quad \varphi_3(w_4)=z_3z_5,$$
$$\varphi_3(w_5)=z_2z_5,\quad \varphi_3(w_6)=z_6,\quad \varphi_3(w_7)=0,\quad \varphi_3(w_8)=-z_7. \tag{6.66}$$

It remains to show that there is no $\mathbb{R}$-algebraic redundance among the generators $w_1+I, w_2+I, w_3+I, w_4+I, w_5+I, w_6+I, w_7+I,$and w_8+I of $\mathfrak{W}_\infty$, where I is the ideal $\langle w_1^2, w_2^2, w_3^2, w_4^2, w_5^2, w_6^2w_7^2, w_8^2, w_1w_5, w_1w_6, w_2w_4, w_2w_6, w_3w_4, w_3w_5, w_4w_5, w_4ww_5w_6, w_1w_7, w_1w_8, w_2w_7, w_2w_8, w_3w_7, w_3w_8, w_4w_7, w_4w_8, w_5w_7, w_5w_8, w_6w_7, w_6w_8, w_7w_8\rangle$. This goes as follows:

(6.67) The reduced Gröbner basis of the ideal $\langle w_1^2, w_2^2, w_3^2, w_2w_4, w_3w_4, w_4^2, w_1w_5, w_3w_5, w_4w_5, w_5^2, w_1w_6, w_2w_6, w_4w_6, w_5w_6, w_6^2, w_1w_7, w_2w_7, w_3w_7, w_4w_7, w_5w_7, w_6w_7, w_7^2, w_1w_8, w_2w_8, w_3w_8, w_4w_8, w_5w_8, w_6w_8, w_7w_8, w_8^2, v_1-w_1, v_2-w_2, v_3-w_3, v_4-w_4, v_5-w_5, v_6-w_6, v_7-w_7, v_8-w_8\rangle$ in the polynomial algebra $\mathbb{R}[w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8]$ with respect to the lexicographic order $w_8>w_7>w_6>w_5>w_4>w_3>w_2>w_1>v_8>v_7>v_6>v_5>v_4>v_3>v_2>v_1$ is $\mathcal{G}_1=\{v_1^2, v_2^2, v_3^2, v_2v_4, v_3v_4, v_4^2, v_1v_5, v_3v_5, v_4v_5, v_5^2, v_1v_6, v_2v_6, v_4v_6, v_5v_6, v_6^2, v_1v_7, v_2v_7, v_3v_7, v_4v_7, v_5v_7, v_6v_7, v_7^2, v_1v_8, v_2v_8, v_3v_8, v_4v_8, v_5v_8, v_6v_8, v_7v_8, v_8^2, v_1-w_1, v_2-w_2, v_3-w_3, v_4-w_4, v_5-w_5, v_6-w_6, v_7-w_7, v_8-w_8\}$, so that $\mathcal{G}_1\cap\mathbb{R}[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8]=\{v_1^2, v_2^2, v_3^2, v_2v_4, v_3v_4, v_4^2, v_1v_5, v_3v_5, v_4v_5, v_5^2, v_1v_6, v_2v_6, v_4v_6, v_5v_6, v_6^2, v_1v_7, v_2v_7, v_3v_7, v_4v_7, v_5v_7, v_6v_7, v_7^2, v_1v_8, v_2v_8, v_3v_8, v_4v_8, v_5v_8, v_6v_8, v_7v_8, v_8^2\}$. This means by Theorem 2.1 that w_8+I is not $\mathbb{R}$-algebraically redundant among $w_1+I, w_2+I, w_3+I, w_4+I, w_5+I, w_6+I, w_7+I$ and w_8+I.

(6.68) The reduced Gröbner basis of the ideal $\langle w_1^2, w_2^2, w_3^2, w_2w_4, w_3w_4, w_4^2, w_1w_5, w_3w_5, w_4w_5, w_5^2, w_1w_6, w_2w_6, w_4w_6, w_5w_6, w_6^2, w_1w_7, w_2w_7, w_3w_7, w_4w_7, w_5w_7, w_6w_7, w_7^2, w_1w_8, w_2w_8, w_3w_8, w_4w_8, w_5w_8, w_6w_8, w_7w_8, w_8^2, v_1-w_2, v_2-w_3, v_3-w_4, v_4-w_5, v_5-w_6, v_6-w_7, v_7-w_8, v_8-w_1\rangle$ in the polynomial algebra $\mathbb{R}[w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8]$ with respect to the lexicographic order $w_8>w_7>w_6>w_5>w_4>w_3>w_2>w_1>v_8>v_7>v_6>v_5>v_4>v_3>v_2>v_1$ is $\mathcal{G}_2=\{v_1^2, v_2^2, v_1v_3, v_2v_3, v_3^2, v_2v_4, v_3v_4, v_4^2, v_1v_5, v_3v_5, v_4v_5, v_5^2, v_1v_6, v_2v_6, v_3v_6, v_4v_6, v_5v_6, v_6^2, v_1v_7, v_2v_7, v_3v_7, v_4v_7, v_5v_7, v_6v_7, v_7^2, v_4v_8, v_5v_8, v_6v_8, v_7v_8, v_8^2, v_8-w_1, v_1-w_2, v_2-w_3, v_3-w_4, v_4-w_5, v_5-w_6, v_6-w_7, v_7-w_8\}$, so that $\mathcal{G}_2\cap\mathbb{R}[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8] = \{v_1^2, v_2^2, v_1v_3, v_2v_3, v_3^2, v_2v_4, v_3v_4, v_4^2, v_1v_5, v_3v_5, v_4v_5, v_5^2, v_1v_6, v_2v_6, v_3v_6, v_4v_6, v_5v_6, v_6^2, v_1v_7, v_2v_7, v_3v_7, v_4v_7, v_5v_7, v_6v_7, v_7^2, v_4v_8, v_5v_8, v_6v_8, v_7v_8, v_8^2\}$. This means by Theorem 2.1 that w_1+I is not $\mathbb{R}$-algebraically redundant among $w_1+I, w_2+I, w_3+I, w_4+I, w_5+I, w_6+I, w_7+I$ and w_8+I.

(6.69) The reduced Gröbner basis of the ideal $\langle w_1^2, w_2^2, w_3^2, w_2w_4, w_3w_4, w_4^2, w_1w_5, w_3w_5, w_4w_5, w_5^2, w_1w_6, w_2w_6, w_4w_6, w_5w_6, w_6^2, w_1w_7, w_2w_7, w_3w_7, w_4w_7, w_5w_7, w_6w_7, w_7^2, w_1w_8, w_2w_8, w_3w_8, w_4w_8, w_5w_8, w_6w_8, w_7w_8, w_8^2, v_1-w_3, v_2-w_4, v_3-w_5, v_4-w_6, v_5-w_7, v_6-w_8, v_7-w_1, v_8-w_2\rangle$ in the polynomial algebra $\mathbb{R}[w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8]$ with respect to the lexicographic order $w_8>w_7>w_6>w_5>w_4>w_3>w_2>w_1>v_8>v_7>v_6>v_5>v_4>v_3>v_2>v_1$ is $\mathcal{G}_3=\{v_1^2, v_1v_2, v_2^2, v_1v_3, v_2v_3, v_3^2, v_2v_4, v_3v_4, v_4^2, v_1v_5, v_2v_5, v_3v_5, v_4v_5, v_5^2, v_1v_6, v_2v_6, v_3v_6, v_4v_6, v_5v_6, v_6^2, v_3v_7, v_4v_7, v_5v_7, v_6v_7, v_7^2, v_2v_8, v_4v_8, v_5v_8, v_6v_8, v_8^2,$

$v_7 - w_1, v_8 - w_2, v_1 - w_3, v_2 - w_4, v_3 - w_5, v_4 - w_6, v_5 - w_7, v_6 - w_8\}$, so that $\mathcal{G}_3 \cap \mathbb{R}[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8] = \{v_1^2, v_1v_2, v_2^2, v_1v_3, v_2v_3, v_3^2, v_2v_4, v_3v_4, v_4^2, v_1v_5, v_2v_5, v_3v_5, v_4v_5, v_5^2, v_1v_6, v_2v_6, v_3v_6, v_4v_6, v_5v_6, v_6^2, v_3v_7, v_4v_7, v_5v_7, v_6v_7, v_7^2, v_2v_8, v_4v_8, v_5v_8, v_6v_8, v_8^2\}$. This means by Theorem 2.1 that $w_2 + I$ is not $\mathbb{R}$-algebraically redundant among $w_1 + I, w_2 + I, w_3 + I, w_4 + I, w_5 + I, w_6 + I, w_7 + I$ and $w_8 + I$.

(6.70) The reduced Gröbner basis of the ideal $\langle w_1^2, w_2^2, w_3^2, w_2w_4, w_3w_4, w_4^2, w_1w_5, w_3w_5, w_4w_5, w_5^2, w_1w_6, w_2w_6, w_4w_6, w_5w_6, w_6^2, w_1w_7, w_2w_7, w_3w_7, w_4w_7, w_5w_7, w_6w_7, w_7^2, w_1w_8, w_2w_8, w_3w_8, w_4w_8, w_5w_8, w_6w_8, w_7w_8, w_8^2, v_1 - w_4, v_2 - w_5, v_3 - w_6, v_4 - w_7, v_5 - w_8, v_6 - w_1, v_7 - w_2, v_8 - w_3\rangle$ in the polynomial algebra $\mathbb{R}[w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8]$ with respect to the lexicographic order $w_8 > w_7 > w_6 > w_5 > w_4 > w_3 > w_2 > w_1 > v_8 > v_7 > v_6 > v_5 > v_4 > v_3 > v_2 > v_1$ is $\mathcal{G}_4 = \{v_1^2, v_1v_2, v_2^2, v_1v_3, v_2v_3, v_3^2, v_1v_4, v_2v_4, v_3v_4, v_4^2, v_1v_5, v_2v_5, v_3v_5, v_4v_5, v_5^2, v_2v_6, v_3v_6, v_4v_6, v_5v_6, v_6^2, v_1v_7, v_3v_7, v_4v_7, v_5v_7, v_7^2, v_1v_8, v_2v_8, v_4v_8, v_5v_8, v_8^2, v_6 - w_1, v_7 - w_2, v_8 - w_3, v_1 - w_4, v_2 - w_5, v_3 - w_6, v_4 - w_7, v_5 - w_8\}$, so that $\mathcal{G}_4 \cap \mathbb{R}[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8] = \{v_1^2, v_1v_2, v_2^2, v_1v_3, v_2v_3, v_3^2, v_1v_4, v_2v_4, v_3v_4, v_4^2, v_1v_5, v_2v_5, v_3v_5, v_4v_5, v_5^2, v_2v_6, v_3v_6, v_4v_6, v_5v_6, v_6^2, v_1v_7, v_3v_7, v_4v_7, v_5v_7, v_7^2, v_1v_8, v_2v_8, v_4v_8, v_5v_8, v_8^2\}$. This means by Theorem 2.1 that $w_3 + I$ is not $\mathbb{R}$-algebraically redundant among $w_1 + I, w_2 + I, w_3 + I, w_4 + I, w_5 + I, w_6 + I, w_7 + I$ and $w_8 + I$.

(6.71) The reduced Gröbner basis of the ideal $\langle w_1^2, w_2^2, w_3^2, w_2w_4, w_3w_4, w_4^2, w_1w_5, w_3w_5, w_4w_5, w_5^2, w_1w_6, w_2w_6, w_4w_6, w_5w_6, w_6^2, w_1w_7, w_2w_7, w_3w_7, w_4w_7, w_5w_7, w_6w_7, w_7^2, w_1w_8, w_2w_8, w_3w_8, w_4w_8, w_5w_8, w_6w_8, w_7w_8, w_8^2, v_1 - w_5, v_2 - w_6, v_3 - w_7, v_4 - w_8, v_5 - w_1, v_6 - w_2, v_7 - w_3, v_8 - w_4\rangle$ in the polynomial algebra $\mathbb{R}[w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8]$ with respect to the lexicographic order $w_8 > w_7 > w_6 > w_5 > w_4 > w_3 > w_2 > w_1 > v_8 > v_7 > v_6 > v_5 > v_4 > v_3 > v_2 > v_1$ is $\mathcal{G}_5 = \{v_1^2, v_1v_2, v_2^2, v_1v_3, v_2v_3, v_3^2, v_1v_4, v_2v_4, v_3v_4, v_4^2, v_1v_5, v_2v_5, v_3v_5, v_4v_5, v_5^2, v_2v_6, v_3v_6, v_4v_6, v_6^2, v_1v_7, v_3v_7, v_4v_7, v_7^2, v_1v_8, v_2v_8, v_3v_8, v_4v_8, v_6v_8, v_7v_8, v_8^2, v_5 - w_1, v_6 - w_2, v_7 - w_3, v_8 - w_4, v_1 - w_5, v_2 - w_6, v_3 - w_7, v_4 - w_8\}$, so that $\mathcal{G}_5 \cap \mathbb{R}[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8] = \{v_1^2, v_1v_2, v_2^2, v_1v_3, v_2v_3, v_3^2, v_1v_4, v_2v_4, v_3v_4, v_4^2, v_1v_5, v_2v_5, v_3v_5, v_4v_5, v_5^2, v_2v_6, v_3v_6, v_4v_6, v_6^2, v_1v_7, v_3v_7, v_4v_7, v_7^2, v_1v_8, v_2v_8, v_3v_8, v_4v_8, v_6v_8, v_7v_8, v_8^2\}$. This means by Theorem 2.1 that $w_4 + I$ is not $\mathbb{R}$-algebraically redundant among $w_1 + I, w_2 + I, w_3 + I, w_4 + I, w_5 + I, w_6 + I, w_7 + I$ and $w_8 + I$.

(6.72) The reduced Gröbner basis of the ideal $\langle w_1^2, w_2^2, w_3^2, w_2w_4, w_3w_4, w_4^2, w_1w_5, w_3w_5, w_4w_5, w_5^2, w_1w_6, w_2w_6, w_4w_6, w_5w_6, w_6^2, w_1w_7, w_2w_7, w_3w_7, w_4w_7, w_5w_7, w_6w_7, w_7^2, w_1w_8, w_2w_8, w_3w_8, w_4w_8, w_5w_8, w_6w_8, w_7w_8, w_8^2, v_1 - w_6, v_2 - w_7, v_3 - w_8, v_4 - w_1, v_5 - w_2, v_6 - w_3, v_7 - w_4, v_8 - w_5\rangle$ in the polynomial algebra $\mathbb{R}[w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8]$ with respect to the lexicographic order $w_8 > w_7 > w_6 > w_5 > w_4 > w_3 > w_2 > w_1 > v_8 > v_7 > v_6 > v_5 > v_4 > v_3 > v_2 > v_1$ is $\mathcal{G}_6 = \{v_1^2, v_1v_2, v_2^2, v_1v_3, v_2v_3, v_3^2, v_1v_4, v_2v_4, v_3v_4, v_4^2, v_1v_5, v_2v_5, v_3v_5, v_5^2, v_2v_6, v_3v_6, v_6^2, v_1v_7, v_2v_7, v_3v_7, v_5v_7, v_6v_7, v_7^2, v_1v_8, v_2v_8, v_3v_8, v_4v_8, v_6v_8, v_7v_8, v_8^2, v_4 - w_1, v_5 - w_2, v_6 - w_3, v_7 - w_4, v_8 - w_5, v_1 - w_6, v_2 - w_7, v_3 - w_8\}$, so that $\mathcal{G}_6 \cap \mathbb{R}[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8] = \{v_1^2, v_1v_2, v_2^2, v_1v_3, v_2v_3, v_3^2, v_1v_4, v_2v_4, v_3v_4, v_4^2, v_1v_5, v_2v_5, v_3v_5, v_5^2, v_2v_6, v_3v_6, v_6^2, v_1v_7, v_2v_7, v_3v_7, v_5v_7, v_6v_7, v_7^2, v_1v_8, v_2v_8, v_3v_8, v_4v_8, v_6v_8, v_7v_8, v_8^2\}$. This means by Theorem 2.1 that $w_5 + I$ is not $\mathbb{R}$-algebraically redundant among $w_1 + I, w_2 + I, w_3 + I, w_4 + I, w_5 + I, w_6 + I, w_7 + I$ and $w_8 + I$.

(6.73) The reduced Gröbner basis of the ideal $\langle w_1^2, w_2^2, w_3^2, w_2w_4, w_3w_4, w_4^2, w_1w_5, w_3w_5, w_4w_5, w_5^2, w_1w_6, w_2w_6, w_4w_6, w_5w_6, w_6^2, w_1w_7, w_2w_7, w_3w_7, w_4w_7, w_5w_7, w_6w_7, w_7^2, w_1w_8, w_2w_8, w_3w_8, w_4w_8, w_5w_8, w_6w_8, w_7w_8, w_8^2, v_1 - w_7, v_2 - w_8, v_3 - w_1,$

$v_4 - w_2, v_5 - w_3, v_6 - w_4, v_7 - w_5, v_8 - w_6\rangle$ in the polynomial algebra $\mathbb{R}[w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8]$ with respect to the lexicographic order $w_8 > w_7 > w_6 > w_5 > w_4 > w_3 > w_2 > w_1 > v_8 > v_7 > v_6 > v_5 > v_4 > v_3 > v_2 > v_1$ is $\mathcal{G}_7 = \{v_1^2, v_1v_2, v_2^2, v_1v_3, v_2v_3, v_3^2, v_1v_4, v_2v_4, v_4^2, v_1v_5, v_2v_5, v_5^2, v_1v_6, v_2v_6, v_4v_6, v_5v_6, v_6^2, v_1v_7, v_2v_7, v_3v_7, v_5v_7, v_6v_7, v_7^2, v_1v_8, v_2v_8, v_3v_8, v_4v_8, v_6v_8, v_7v_8, v_8^2, v_3 - w_1, v_4 - w_2, v_5 - w_3, v_6 - w_4, v_7 - w_5, v_8 - w_6, v_1 - w_7, v_2 - w_8\}$, so that $\mathcal{G}_7 \cap \mathbb{R}[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8] = \{v_1^2, v_1v_2, v_2^2, v_1v_3, v_2v_3, v_3^2, v_1v_4, v_2v_4, v_4^2, v_1v_5, v_2v_5, v_5^2, v_1v_6, v_2v_6, v_4v_6, v_5v_6, v_6^2, v_1v_7, v_2v_7, v_3v_7, v_5v_7, v_6v_7, v_7^2, v_1v_8, v_2v_8, v_3v_8, v_4v_8, v_6v_8, v_7v_8, v_8^2\}$. This means by Theorem 2.1 that $w_6 + I$ is not $\mathbb{R}$-algebraically redundant among $w_1 + I, w_2 + I, w_3 + I, w_4 + I, w_5 + I, w_6 + I, w_7 + I$ and $w_8 + I$.

(6.74) The reduced Gröbner basis of the ideal $\langle w_1^2, w_2^2, w_3^2, w_2w_4, w_3w_4, w_4^2, w_1w_5, w_3w_5, w_4w_5, w_5^2, w_1w_6, w_2w_6, w_4w_6, w_5w_6, w_6^2, w_1w_7, w_2w_7, w_3w_7, w_4w_7, w_5w_7, w_6w_7, w_7^2, w_1w_8, w_2w_8, w_3w_8, w_4w_8, w_5w_8, w_6w_8, w_7w_8, w_8^2, v_1 - w_8, v_2 - w_1, v_3 - w_2, v_4 - w_3, v_5 - w_4, v_6 - w_5, v_7 - w_6, v_8 - w_7\rangle$ in the polynomial algebra $\mathbb{R}[w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8]$ with respect to the lexicographic order $w_8 > w_7 > w_6 > w_5 > w_4 > w_3 > w_2 > w_1 > v_8 > v_7 > v_6 > v_5 > v_4 > v_3 > v_2 > v_1$ is $\mathcal{G}_8 = \{v_1^2, v_1v_2, v_2^2, v_1v_3, v_3^2, v_1v_4, v_4^2, v_1v_5, v_3v_5, v_4v_5, v_5^2, v_1v_6, v_2v_6, v_4v_6, v_5v_6, v_6^2, v_1v_7, v_2v_7, v_3v_7, v_5v_7, v_6v_7, v_7^2, v_1v_8, v_2v_8, v_3v_8, v_4v_8, v_5v_8, v_6v_8, v_7v_8, v_8^2, v_2 - w_1, v_3 - w_2, v_4 - w_3, v_5 - w_4, v_6 - w_5, v_7 - w_6, v_8 - w_7, v_1 - w_8\}$, so that $\mathcal{G}_8 \cap \mathbb{R}[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8] = \{v_1^2, v_1v_2, v_2^2, v_1v_3, v_3^2, v_1v_4, v_4^2, v_1v_5, v_3v_5, v_4v_5, v_5^2, v_1v_6, v_2v_6, v_4v_6, v_5v_6, v_6^2, v_1v_7, v_2v_7, v_3v_7, v_5v_7, v_6v_7, v_7^2, v_1v_8, v_2v_8, v_3v_8, v_4v_8, v_5v_8, v_6v_8, v_7v_8, v_8^2\}$. This means by Theorem 2.1 that $w_7 + I$ is not $\mathbb{R}$-algebraically redundant among $w_1 + I, w_2 + I, w_3 + I, w_4 + I, w_5 + I, w_6 + I, w_7 + I$ and $w_8 + I$.

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